International Journal of Engineering Inventions
e-ISSN: 2278-7461, p-ISSN: 2319-6491
Volume 3, Issue 7 (February 2014) PP: 09-20

Exploring the dynamics of labour market through bifurcation theory

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Abstract: *This paper provides a modeling framework for the microeconomic and macroeconomic analysis of labor market equilibrium. The study explores the work of Yuri Kuznetsov and Ronald Shone regarding the dynamics of unemployment (and employment) by providing two different settings for the matching rate m(u,v). The aim of this paper is to develop dynamic models in order to explain a set of facts regarding job flows. The stability of the central variety for the new generated ode systems is analyzed by using the software package of xpp.exe. The main results of this research are the theorems on the existence and location of Hopf bifurcation boundaries in each of the considered cases.*
Key Words: *Hopf bifurcation, equilibrium points, search and matching, open economy, labour market.*
2010 MSC: 34C23, 37G15

I. Introduction

Despite some scattered signs of improvement, the world economic situation is still facing a serious jobs challenge, with global unemployment above its pre-crisis level and unemployment in the euro area rising rapidly. Most developed economies are still struggling to overcome four major issues: the continued deleveraging by banks, firms and households holding back normal credit flows and consumer and investment demand, high unemployment which is both cause and effect of the lack of economic recovery, third, the fiscal austerity responses to deal with rising public debts and fourth, bank exposure to sovereign debts, the financial sector fragility (World Economic Situation and Prospects, 2012). Due to worsening economic conditions and drastic spending cuts, unemployment in the euro area rises to record levels and it is expected to stabilize and gradually decline as growth recovers (The Current Economic Situation - Euro Challenge, March 2012). A long standing challenge in macroeconomics is developing a unique and comprehensive model which could explain all relevant observations to the level and evolution of unemployment. Since the early 1960, the interest in dynamical characteristics of unemployment has been rapidly growing. This study aims to develop dynamic models in order to explain a set of facts regarding job flows, unemployment and output. The paper is organized as follows: after a brief introduction, the second section brings evidence on the work of Yuri Kuznetsov and Ronald Shone regarding the dynamics of unemployment (and employment). The core of dynamics is bifurcation theory which is fundamental to systems theory.

Section 3 introduces basic definitions and theorems of bifurcation theory employed in this paper, Hopf bifurcation being the most commonly seen type of bifurcation since the existence of its boundary is accompanied by regular oscillations when the parameters are within a neighborhood of the boundary, and where the oscillations may damp to a stable steady state or may never damp depending upon the side of the bifurcation boundary on which the parameter might lie (William A. Barnett et all., 2008). As within the development of the theory, the matching function has been widely used to help model the inefficiencies in labor markets, further on, the study extends the framework along with the findings of its most proeminent theorists, Diamond, Mortensen and Pissarides.

Section 4 extends Ronald Shone's approach to the dynamics of unemployment by providing two different settings for the matching rate m(u,v). Propositions establishing the existence of Hopf bifurcation are also formulated and proved for each case. This section also lays out their phase diagrams and their economic application. Central results in this research are the theorems on the existence and location of Hopf bifurcation boundaries in each of the considered cases. The main results of the research are summarized in section 5.

II. The Basic Model

In order to study the dynamics of labour market, this research is based on the model developed by Ronald Shone (2002), who considers that at the ruling wage there is full employment in the sense that the number of jobs is matched by the number of households seeking employment. The working population population, *N*, is considered to be fixed and the number of jobs available to be constant. At any instant of time a fraction s of individuals become unemployed and search over firms to find a suitable job. Let f denote the probability of finding a job, i.e., the fraction finding a job. At any moment of time, if u is the fraction of the participating labour force unemployed, then

$s(1-u)N =$ individuals entering the unemploymet pool
$fuN =$ individuals exiting the unemployment pool

The change in the unemployment pool, $uN,$ is therefore given by the differential equation

$$\frac{d(uN)}{dt} = s(1-u)N - fuN \qquad 0 < s < 1,\ \ 0 < f < 1$$

Since N is constant then

$$(*) \qquad \dot{u} = \frac{du}{dt} = s(1-u) - fu$$

or

$$\dot{u} = s - (s+f)u$$

The Equilibrium requires that $\frac{du}{dt} = 0,$ or

$$s - (s+f)u^* = 0$$

i.e.,

$$u^* = \frac{s}{s+f} = \frac{s/f}{1+(s/f)}$$

where

$$\frac{\partial u^*}{\partial s} = \frac{f}{(s+f)^2} > 0, \qquad \frac{\partial u^*}{\partial s} = \frac{-s}{(s+f)^2} < 0$$

In other words, the equilibrium unemployment rate- the natural rate in this model- rises as more individuals enter the unemployment pool to actively search for a job and falls when the job-finding rate becomes greater. But this simple model says more than this about the equilibrium (natural) level of unemployment. It says that the level of u^* occurs because individuals need to seek alternative unemployment and that the search for a new jobs takes time.

The time path is readily found by solving the differential equation $(*)$. If $u(0) = u_0,$ then

$$u(t) = u^* + (u_0 - u^*)e^{-(s+f)t}, \quad u^* = \frac{s}{s+f}$$

Since both s and f are positive, then this solution implies that unemployment tends to its equilibrium value over time.

In this model, the focus is on the level of *unemployment*. Of course, if N is fixed, then the employment, $E,$ is simply

$$E = (1-u)N$$

or

$$e = E/N = (1-u),$$

where e is the employment rate. In order to lay the foundation for other dynamic theories, it is worth noting that at any moment of time there will be an unemployment rate of $u = U/N,$ and a vacancy rate of $v = V/N.$ Since N is constant throughout, we can concentrate on the rates u, v and $e.$

At any moment of time there will be an unemployment rate u and a vacancy rate $v,$ where those individuals who are unemployed are attempting to match themselves with the available vacancies. Since we have assumed that the number of jobs is matched by the number of those seeking employment, then $u = v.$, the problem is one of matching the unemployment to the vacancies. Accordingly, the literature refers to the matching rate or the *exchange technology* (Mortensen, 1990) [23]. In other words, the unemployment and the jobs that employers are seeking to fill are *inputs* into the meeting process. Let this be denoted $m(u,v).$

Given $m(u,v),$ then for such a meeting to take place must either be some unemployment or some vacancies. More formally $m(0,v) = m(u,0) = 0.$ Furthermore, the marginal contribution of each *input* is positive,i.e., $\partial m/\partial u > 0$ and $\partial m/\partial v > 0.$ Following Diamond 1982 [14], it is further assumed that the average return to each *input* is diminishing, i.e., m/u and m/v diminishes with u and $v,$ respectively. Finally, and purely for mathematical convenience, we assume that $m(u,v)$ is homogeneous of degree $k,$ so that

$$m(u,v) = u^k m(1, v/u)$$

Using this analysis we can write the change in employment as the total match $Nm(u,v)$ minus those losing a job $s(1-u)N,$ i.e.

$$\frac{\partial E}{\partial t} = \frac{\partial (eN)}{\partial t} = Nm(u,v) - s(1-u)N$$

or

$$\frac{\partial e}{\partial t} = \dot{e} = m(u,v) - se$$

Although the time path of employment, $e(t)$, must mirror the time path of the unemployment rate , $u(t)$, since $e = 1-u$, the present formulation directs attention to the matching rate $m(u,v)$.

In general (Mortensen 1990, [23]), the equilibrium hiring frequency, $m(u,v)/u$, is a function of the present value of employment per worker to the firm ,q, and the employment rate , e. This can be established by noting that

$$(**) \qquad \frac{m(u,v)}{u} = \frac{u^k m(1,v/u)}{u} = u^{k-1}m(1,v/u) = (1-e)^{k-1}m(1,v/u) = h(q,e).$$

The hiring function $h(q,e)$ is a function of q since the value of v/u in $(**)$ is determined in equilibrium. In equilibrium, the return on filling a vacancy (mq/v) is equal to the cost of filling vacancy, c, i.e.,

$$[\frac{m(u,v)}{v}]q = c$$

that is

$$(1-e)^{k-1}q = \frac{cv/u}{m(1,v/u)}$$

so, the hiring frequency is related to both q and e. Furthermore, we can establish from this last result that $h_q > 0$ and $h_e < 0$ if $k > 1$ and $h_e > 0$ if $k < 1$. Hence

$$\frac{m(u,v)}{u} = h(q,e) \qquad h_q > 0,$$

System 1

$$\begin{cases} h_e < 0 \;\; if \;\; k > 1 \\ h_e > 0 \;\; if \;\; k < 1. \end{cases}$$

$$m(u,v) = uh(q,e) = (1-e)h(q,e)$$

which in turn leads to the following equilibrium adjustment equation

$$\dot{e} = (1-e)h(q,e) - se$$

The profit to the firm of hiring an additional worker is related to q and the employment rate e, e.i, $\pi(q,e)$, and it will be different for different models of the labour market. The profit arises from the difference in the marginal revenue product per worker, MRP_L, less the paid wage, w. If we denote the MRP_L by $x(e)$, then $\pi(q,e) = x(e) - w$. However, the future profit stream per worker to the firm is

$$rq = x(e) - w - s(q - k_v) + \dot{q}$$

where rq represents the opportunity interest in having a filled vacancy and k_v is the capital value of a vacant job ,i.e., the present value of employment to the firm is the profit from hiring the worker less the loss from someone becoming unemployed plus any capital gain.

Since in equilibrium no vacancies exist, then $k_v = 0$ and so

$$rq = \pi(q,e) - sq + \dot{q}$$
$$\dot{q} = (r+s)q - \pi(q,e)$$

To summarise , we have two differential equations in e and q i.e.,

System 2

$$\begin{cases} \dot{e} = (1-e)h(q,e) - se \\ \dot{q} = (r+s)q - \pi(q,e) \end{cases}$$

Whether there is a unique equilibrium, it rests very much on the degree of homogeneity of the match function i.e., the value of k in $m(u,v) = u^k m(1,\frac{v}{u})$ and the productivity per worker $x(e)$.

III. Bifurcation Phenomena in Economics Literature

Over the last three decades, the theoretical research in macroeconomics has moved from comparative statics to dynamics, with many such dynamical models exhibiting nonlinear dynamics. Bifurcation analysis is a key tool for the analysis of dynamic systems in general and nonlinear systems in particular. It has been widely used in mathematics and engineering and although it is a relatively new in economics, the interest for this type of analysis has been increasing because it provides information about the occurrence and changes in stability of fixed points, limit cycles and other solution paths, it helps model these changes and transitions from stable to unstable case or vice versa as some parameters change. Most common types of bifurcations encountered in economic analysis include Saddle Node, Transcritical, Pitchfork, Flip and Hopf bifurcations (William A. Barnett, 2008). Each of these bifurcations produces a different type of qualitative change in dynamics.

The first paper on Hopf bifurcation belongs to $Poincare'$ (1892) and the first formulation of the theorem to Andronov (1929). Another important theorem on the existence of Hopf bifurcation appeared in Hopf (1942). Its existence has been proved in many economic models. For example, Torre (1977), observed the existence of a limit cycle associated with a Hopf bifurcation boundary in a study on Keynesian systems. Benhabib and Nishimura (1979) explored a multi-sector neoclassical optimal growth model and proved that a closed invariant curve might be a result of optimization. Hopf bifurcations were also found in other studies such as Aiyagari (1989), Benhabib and Day (1982), Benhabib and Rustichini (1991), Gale (1973). Recent findings belong to Barnett and He (1999, 2001, 2002, 2004, 2006, 2008), who found bifurcation boundaries in a Bergstrom continuous-time model of the UK economy and to Leeper and Sims with their Euler-equations model of the United States economy.

Barnett and Duzhak (2008, 2010) analyzed the bifurcation phenomenon using a closed economy New Keynesian model and they found both Hopf and Period Doubling bifurcations. Despite the growing interest in bifurcation analysis of economic systems, the literature on this subject is still immature and needs an extensive study for a comprehensive understanding. The present paper studies the existence of Hopf bifurcation for the model described by Ronald Shone (2002), by considering different settings for the matching function denoted $m(u,v)$.

A Preliminary Review of Hopf Bifurcation Theory

According to Yuri Kuznetsov (2004), if we consider the following system of two differential equations depending on one parameters.

Theorem 1 *[20] Suppose a two dimensional system*

$$\frac{dx}{dt} = f(x,\alpha),\ x \in \mathbf{R^2},\ \alpha \in \mathbf{R^1}, \qquad (1.1)$$

with smooth f , has for all sufficiently small $|\alpha|$ the equilibrium $x = 0$ with eigenvalues

$$\lambda_{1,2}(\alpha) = \mu(\alpha) + i\omega(\alpha),$$

where $\mu(0) = 0,\ \omega(0) = \omega_0 > 0.$

Let the following conditions be satisfied:

(B.1) $l_1(0) \neq 0,$ where l_1 is the first Lyapunov coefficient;

(B.2) $\mu^{'}(0) \neq 0.$

Then, there are invertible coordinate and parameter changes and a time reparameterization transforming (1.1) into

$$\frac{d}{d\tau}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & -\beta \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm ({y_1}^2 + {y_2}^2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + O(||y||^4).$$

We can drop the $O(||y||^4)$ terms and finally arrive at the following general result.

Theorem 2 *([20] Topological normal form for the Hopf bifurcation)*

Any generic two-dimensional, one-parameter system

$$\dot{x} = f(x,\alpha),$$

having at $\alpha = 0$ the equilibrium $x = 0$ with eigenvalues

$$\lambda_{1,2}(\alpha) = \pm i\omega_0,\ \omega_0 > 0,$$

is locally topologically equivalent near the origin to one of the following normal forms:

$$\frac{d}{d\tau}\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ 1 & -\beta \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \pm ({y_1}^2 + {y_2}^2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Consider the following system of two differential equations depending on one parameter:

System 3

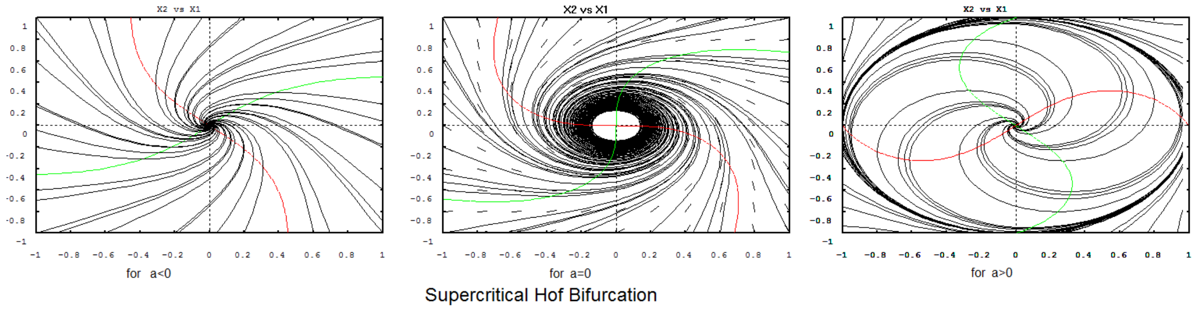
$$\begin{cases} \dot{x}_1 = \alpha x_1 - x_2 + s \cdot x_1({x_1}^2 + {x_2}^2) \\ \dot{x}_2 = x_1 + \alpha x_2 + s \cdot x_2({x_1}^2 + {x_2}^2). \end{cases}$$

This system has three equilibriums $x_1 = x_2 = 0$ for all α with the Jacobian matrix

$$A = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}$$

having eigenvalues $\lambda_{1,2} = \alpha \pm i$.

Figure 1 .Supercritical Hopf bifurcation

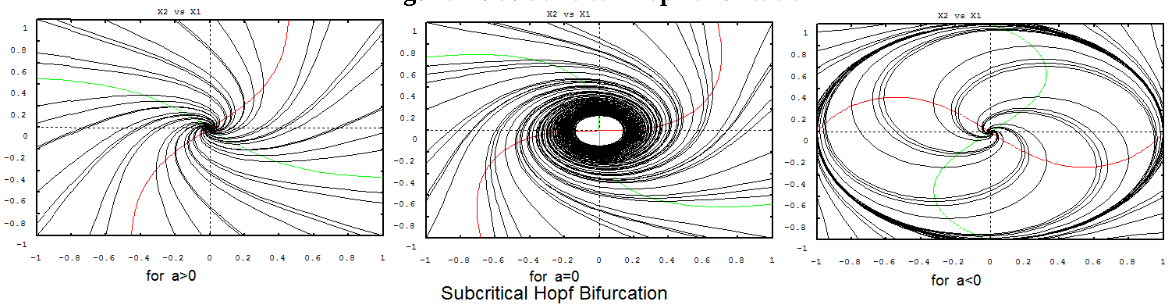


Source: authors' estimations

For $s = -1$, System 3 always has an equilibrium point at the origin. This equilibrium is a stable focus for $\alpha < 0$ and an unstable focus for $\alpha > 0$. At the critical parameter value $\alpha = 0$ the equilibrium is nonlinearly stable and topologically equivalent to the focus. Some times it is called a *weakly attracting focus.* This equilibrium is surrounded for $\alpha > 0$ by an isolated closed orbit *(limit cycle)* that is unique and stable.

The cycle is a circle of radius $\rho_0(\alpha) = \sqrt{\alpha}$. All orbits starting outside or inside the cycle except for the origin tend to the cycle as $t \to +\infty$. This is an Supercritical Andronov-Hopf bifurcation, this being represented in Figure 1.

Figure 2 . Subcritical Hopf bifurcation



Source: authors' estimations

For $s = 1$ in system 3 we have a Subcritical Hopf Bifurcation (presented in Figure 2).

The system undergoes the Andronov-Hopf bifurcation at $\alpha = 0$. Contrary to System 3 for $s = -1$, there is an unstable limit cycle in 3, which disappears when α crosses zero from negative to positive values. The equilibrium at the origin has the same stability for $\alpha \neq 0$ as in System 3 for $s = -1$: It is stable for $\alpha < 0$ and unstable for $\alpha > 0$. Its stability at the critical parameter value is opposite to that in 3: It is (nonlinearly) unstable at $\alpha = 0$.

Remark 1 We have seen that there are two types of Andronov-Hopf bifurcation. The bifurcation in System 3, for $s = -1$, is often called supercritical because the cycle exists for positive value of parameter α ("after" the bifurcation).

The bifurcation in System 3, for $s = 1$ is called *subcritical* since the cycle is present "before" the bifurcation. It is clear that this terminology is somehow misleading since "after" and "before" depend on the chosen direction of parameter variation.

IV. Examples of Dynamical Systems and their Stability Analysis

This section develops and analyses two dynamical systems resulting from using, in System 2, the following expressions for the matching rate.

$$\begin{cases} m_1(u,v) = \alpha \cdot \sigma_m u^\sigma v^\sigma (u^\sigma + v^\sigma) \\ m_2(u,v) = \sigma_m u^\sigma v^{1-\sigma} \end{cases}$$

where σ is the elasticity of the matches with respect to unemployment but also the elasticity of the vacancy filling rate with respect to the labor market tightness , σ_m represents the efficiency of the matching process and α is a scale parameter. The first setting for the matching rate when $m_1(u,v) = uv(u+v)$ was proposed and analysed by L. C. Holdon (2011). We extend that function $m_1(u,v)$ to a more general one, where $\alpha \in (0,1]$ is a scale parameter which can be used to particularize every economy, and we keep the linear function $\pi(q,e) := q \cdot e$.

The second setting is widely used in economics to help model the inefficiencies that are seen in many markets where two agents seek out each other in order to come to an agreement (search frictions). Though, its main application is within the labour market.

The matching function theory has helped to improve models related to business cycles and has become the most significant and important tool in analyzing the labour market, particulary with regards to macroeconomy. It is used by a variety of governments when trying to decide upon unemployment policy, and macroeconomic policy as a whole , its simplified implication being that the lower the level of search friction, the more efficient the relevant market is, and as such the lower the cost that is associated with the relevant markets pairings.
The greatest succes and research in this sense belongs to Diamond, Mortensen and Pissarides, whose model is the most prominent macroeconomic tool which shows the relationship between the rate at which the unemployed are hired with the number of people looking for jobs, and the volume of vacancies available.
Further on, the study will explore the ode systems from a mathematical view point.

From introduction we know that $u = 1 - e$ and denote $q := \frac{v}{u}$.

Namely,
$$\begin{cases} e := x \in (0,1) \\ q := y \in \mathbf{R} \end{cases}$$

and by equation $(**)$ we have

$$\begin{cases} m_1(u,v) = \alpha \cdot \sigma_m (1-x)^{3\sigma} y^\sigma (1 + y^\sigma) \\ m_2(u,v) = \sigma_m (1-x) \cdot y^{1-\sigma}. \end{cases}$$

Following System 2, we obtain the o.d.e systems of:

System 4

$$\begin{cases} \dot{x} = \alpha \cdot \sigma_m (1-x)^{3\sigma} y^\sigma (1 + y^\sigma)) - s \cdot x \quad := f^1(x,y) \\ \dot{y} = (r+s) \cdot y - x \cdot y \quad := f^2(x,y) \end{cases}$$

System 5

$$\begin{cases} \dot{x} = \sigma_m (1-x) \cdot y^{1-\sigma} - s \cdot x \quad := g^1(x,y) \\ \dot{y} = (r+s) \cdot y - x \cdot y \quad := g^2(x,y) \end{cases}$$

where (r,s) are parameters, with $s \in (0,1)$ and $r \in \mathbb{R}$;
x is the *employment rate* and y represents the *labour market tightness*.

The Study of System 4

Applying the continuous transformation $(x, y^\sigma) \longrightarrow (x, \tilde{y})$ we obtain for the first equation of System 4:

$$\dot{x} = \alpha \cdot \sigma_m (1-x)^{3\sigma} \tilde{y}(1+\tilde{y}) - s \cdot x,$$

and for the second equation we obtain:

$$\dot{\tilde{y}} = \sigma \cdot y^{\sigma-1} \cdot \dot{y} = \sigma \cdot y^{\sigma-1} \cdot [(r+s) \cdot y - x \cdot y] = \sigma \cdot [(r+s) \cdot y^\sigma - x \cdot y^\sigma] = \sigma[(r+s) \cdot \tilde{y} - x \cdot \tilde{y}].$$

Therefore, the equivalent ode system with System 4 is:

System 6

$$\begin{cases}\dot{x} = \alpha \cdot \sigma_m (1-x)^{3\sigma} \tilde{y}(1+\tilde{y}) - s \cdot x & := f^1(x,\tilde{y}) \\ \dot{\tilde{y}} = \sigma[(r+s)\cdot \tilde{y} - x \cdot \tilde{y}] & := f^2(x,\tilde{y})\end{cases}$$

For System 6, we have three equilibrium points:

$$E_0 = (0{,}0),\ E_1 = (r+s, -\frac{1}{2} - \frac{1}{2} \cdot \sqrt{1 + \frac{4s(r+s)}{\alpha \cdot \sigma_m (1-r-s)^{3\sigma}}})$$

and

$$E_2 = (r+s, -\frac{1}{2} + \frac{1}{2} \cdot \sqrt{1 + \frac{4s(r+s)}{\alpha \cdot \sigma_m (1-r-s)^{3\sigma}}}).$$

The Jacobian matrix associated to system 4 is:

$$J := \begin{pmatrix} -3 \cdot \alpha \cdot \sigma \cdot \sigma_m \cdot (1-\overline{x})^{(3\sigma-1)} \cdot \overline{\tilde{y}}(1+\overline{\tilde{y}}) - s & \alpha \cdot \sigma_m \cdot (1-\overline{x})^{3\sigma}(1+2\overline{\tilde{y}}) \\ -\sigma\overline{\tilde{y}} & \sigma[-\overline{x} + (r+s)] \end{pmatrix}$$

where $E = (\overline{x}, \overline{\tilde{y}})$ is an equilibrium point.

Since $s \in (0{,}1)$, we have:

Proposition 1 *The equilibrium point E_0 is:*
(i) an unstable saddle point for $r+s > 0$.
(ii) an attractive stable node for $r+s < 0$.
(iii) a nonlinearly attractive equilibrium by type saddle-node for $r+s = 0$.

The Characteristic Equation associated to point E_0 is:

$$\lambda^2 - (-s + \sigma \cdot (r+s))\lambda + (-s) \cdot \sigma \cdot (r+s) = 0,$$

with

$$\Delta_\lambda = (s + \sigma(r+s))^2 \Rightarrow \lambda_1 = \sigma \cdot (r+s)\ \ and\ \ \lambda_2 = -s,$$

because $0 < s < 1 \Rightarrow \lambda_2 < 0$ for all s.
For the equilibrium E_1 we obtain the following:

$$TrJ_1 = -3 \cdot \sigma \cdot \frac{s(r+s)}{1-r-s} - s$$

and

$$DetJ_1 = \frac{1}{2} \cdot \alpha \cdot \sigma \cdot \sigma_m \cdot (1-r-s)^{3\sigma} \sqrt{1 + \frac{4s(r+s)}{\alpha \cdot \sigma_m (1-r-s)^{3\sigma}}} (1 + \sqrt{1 + \frac{4s(r+s)}{\alpha \cdot \sigma_m (1-r-s)^{3\sigma}}}).$$

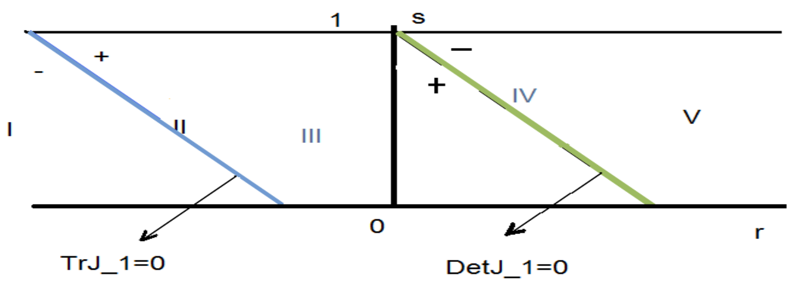
with its characteristic equation:

$$\lambda^2 - (-3 \cdot \sigma \cdot \frac{s(r+s)}{1-r-s} - s)\lambda + (\frac{1}{2} \cdot \alpha \cdot \sigma \cdot \sigma_m \cdot (1-r-s)^{3\sigma} \sqrt{1 + \frac{4s(r+s)}{\alpha \cdot \sigma_m (1-r-s)^{3\sigma}}} (1 + \sqrt{1 + \frac{4s(r+s)}{\alpha \cdot \sigma_m (1-r-s)^{3\sigma}}})) = 0.$$

Denoting:

$$\beta := \sqrt{1 + \frac{4s(r+s)}{\alpha \cdot \sigma_m (1-r-s)^{3\sigma}}} \Rightarrow DetJ_1 = \frac{1}{2} \cdot \alpha \cdot \sigma \cdot \sigma_m \cdot (1-r-s)^{3\sigma} \cdot \beta \cdot (1+\beta).$$

Figure 3. Parameter portrait for System 6 for the equilibrium points E_1



Source: authors' figure

Proposition 2 *The equilibrium point E_1 (see Figure3) is:*

(i) For zone I, it is a stable attractive node when $\lambda_{1,2} \in \mathbf{R}_-$, and stable attractive focus when $\lambda_{1,2} \notin \mathbf{R}$, with $Re\lambda_{1,2} < 0$.

(ii) For zone II, it is a nonlinearly point of Hopf type.

(iii) For zone III, it is an unstable repulsive node when $\lambda_{1,2} \in \mathbf{R}_+$, and an unstable repulsive focus when $\lambda_{1,2} \notin \mathbf{R}$, with $Re\lambda_{1,2} > 0$.

(iv) For zone IV, it is a nonlinearly equilibrium by type saddle-node.

(v) For zone V, it is an unstable saddle.

From mathematical view point, the next result is the most important, it describes the existence of Hopf bifurcation in System 6, with serious repercussions for a country's economy.

Theorem 3 *If* $r + s + \frac{1}{3\cdot\sigma-1} = 0$ *and* $DetJ_1 > 0$, *corresponds to zone II. Then the equilibrium point* E_1 *is a nonlinearly equilibrium point of Hopf type.*

Proof. We must verify the two conditions from Theorem 1.

First, we must calculate $l_1\left(r + s + \frac{1}{3\cdot\sigma-1} = 0\right)$, which is the first Lyapunov coefficient, when $r + s + \frac{1}{3\cdot\sigma-1} = 0$. For obtaining $l_1(r + s + \frac{1}{3\cdot\sigma-1} = 0)$, we will use the formula from [18]:

$$l_1(x,y)|_{(r+s+\frac{1}{3\sigma-1}=0)} := \frac{1}{16}[f^1_{xxx} + f^1_{xyy} + f^2_{xxy} + f^2_{yyy}] + \frac{1}{\omega_0}\frac{1}{16}[f^1_{xy}(f^1_{xx} + f^1_{yy}) - f^2_{xy}(f^2_{xx} + f^2_{yy}) - f^1_{xx}f^2_{xx} + f^1_{yy}f^2_{yy}],$$

where

System 7

$$\begin{cases} \dot{x} = \alpha\cdot\sigma_m(1-x)^{3\sigma}\tilde{y}(1+\tilde{y}) - s\cdot x \quad := f^1(x,\tilde{y}) \\ \dot{\tilde{y}} = \sigma[(r+s)\cdot\tilde{y} - x\cdot\tilde{y}] \quad := f^2(x,\tilde{y}) \end{cases}$$

and $f^1_{xxx} := \frac{\partial^3 f^1(x,y)}{\partial x^3}$.

Since $\sigma_m, \sigma, s > 0$ and $r + s + \frac{1}{3\cdot\sigma-1} = 0 \Rightarrow r + s \neq 1 \Rightarrow \beta > 0 \Rightarrow$

$$\omega_0 = 2\cdot\sqrt{\frac{1}{2}\cdot\alpha\cdot\sigma_m\cdot\sigma\cdot(1-r-s)^{3\sigma}\cdot\beta\cdot(1+\beta)} > 0$$

Since $f^2_{xx} = f^2_{yy} = 0$ we have

$$l_1(x,y)|_{(r+s+\frac{1}{3\cdot\sigma-1}=0)} = \frac{1}{16}[f^1_{xxx} + f^1_{xyy}] + \frac{1}{\omega_0}\frac{1}{16}f^1_{xy}(f^1_{xx} + f^1_{yy}),$$

hence

$$l_1(x,y)|_{\left(r+s+\frac{1}{3\cdot\sigma-1}=0\right)} = (-1)\cdot\{\frac{1}{16}\cdot\alpha\cdot[3\cdot\sigma_m\cdot\sigma\cdot(3\sigma-1)\cdot(3\sigma-2)(1-x)^{3(\sigma-1)}\tilde{y}(1+\tilde{y}) + \quad 6\cdot\sigma\cdot\sigma_m\cdot(1-x)^{(3\sigma-1)}] + \frac{1}{\omega_0}\frac{1}{16}3\cdot\alpha^2\cdot\sigma\cdot\sigma_m\cdot(1-x)^{(3\sigma-1)}(1+2\tilde{y})\cdot[3\sigma\cdot\sigma_m\cdot(3\sigma-1)(1-x)^{(3\sigma-2)}\tilde{y}(1+\tilde{y}) + 2\cdot\sigma_m\cdot(1-x)^{3\sigma}]\}.$$

We recall that $x = r + s = \frac{1}{1-3\cdot\sigma} \neq 1$ and $\sigma_m, \sigma > 0$, so, we deduce that

$$l_1\left(r+s+\frac{1}{3\cdot\sigma-1}=0\right) \neq 0\ , \qquad for\ \ 0 < s < 1.$$

And the second condition from Theorem 1, since $r+s+\frac{1}{3\cdot\sigma-1} = 0$ and $s \in (0,1)$::

$$\frac{\partial\left(r+s+\frac{1}{3\cdot\sigma-1}\right)}{\partial s} = 1 \neq 0;\ \ \frac{\partial\left(r+s+\frac{1}{3\cdot\sigma-1}\right)}{\partial r} = 1 \neq 0.$$

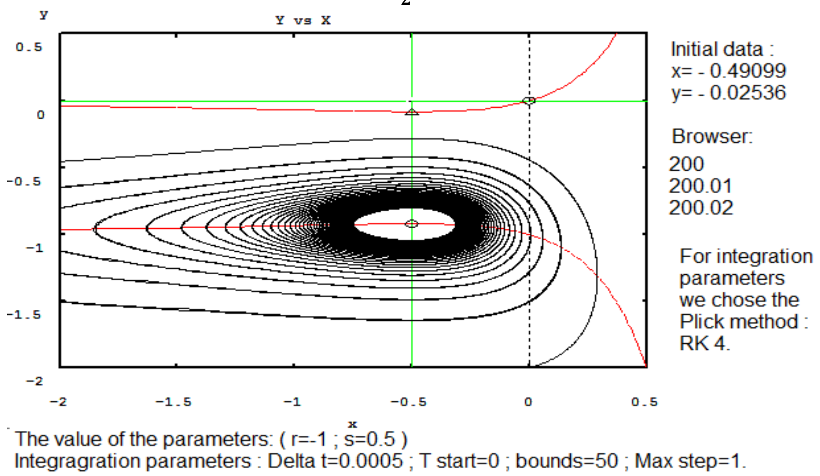
Following Theorem 1 we deduce the existence of Hopf bifurcation.
If inside of Theorem 3 we consider $\alpha = \sigma = \sigma_m = 1$ then we obtain the next result, presented in Holdon(2011) (see [22]), with a clear picture which illustrates the Hopf bifurcation for $r+s+\frac{1}{2} = 0$, associated to the system:

System 8

$$\begin{cases}\dot{x} = (1-x)^3\cdot y\cdot(1+y) - s\cdot x \quad := f^1(x,y)\\ \dot{y} = (r+s)\cdot y - x\cdot y \quad := f^2(x,y)\end{cases}$$

Corollary 1 *If* $r+s+\frac{1}{2} = 0$ *and* $DetJ_1 > 0$, *(in System 8), corresponds to zone II. Then the equilibrium point* E_1 *is a nonlinearly equilibrium point of Hopf type, see Figure 4, and the Hopf bifurcation is subcritical.*

Figure 4: $\mathbf{r+s+\frac{1}{2}=0}$, see [22]



Source: authors' figure

For the equilibriums E_2 we obtain the following:

$$TrJ_2 = -3\cdot\sigma\cdot\frac{s(r+s)}{1-r-s} - s$$

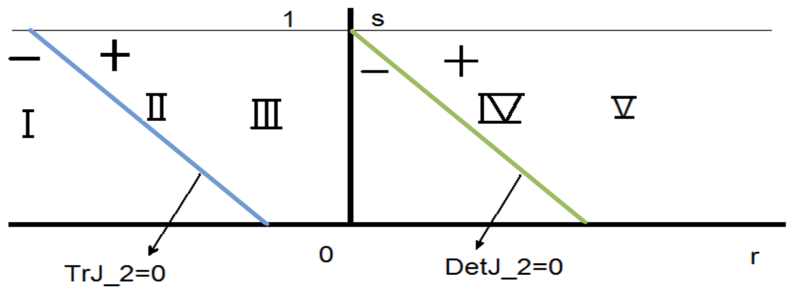
and

$$DetJ_2 = -\frac{1}{2}\cdot\alpha\cdot\sigma\cdot\sigma_m\cdot(1-r-s)^{3\sigma}\sqrt{1+\frac{4s(r+s)}{\alpha\cdot\sigma_m(1-r-s)^{3\sigma}}}\left(1-\sqrt{1+\frac{4s(r+s)}{\alpha\cdot\sigma_m(1-r-s)^{3\sigma}}}\right).$$

with its characteristic equation: $\lambda^2 - \left(-3\cdot\sigma\cdot\frac{s(r+s)}{1-r-s} - s\right)\lambda + \left(-\frac{1}{2}\cdot\alpha\cdot\sigma\cdot\sigma_m\cdot(1-r-\right.$

$s)^{3\sigma}\sqrt{1+\frac{4s(r+s)}{\alpha\cdot\sigma_m(1-r-s)^{3\sigma}}}(1-\sqrt{1+\frac{4s(r+s)}{\alpha\cdot\sigma_m(1-r-s)^{3\sigma}}})) = 0.$

Figure 5. Parameter portrait of system 6 for the equilibrium point E_2



Source: authors' figure

Proposition 3 *The equilibrium point E_2 (see Figure 5) is:*
(i) For zone I, it is an unstable saddle.
(ii) For zone II, it has the eigenvalues

$$\lambda_{1,2} = \pm_2\sqrt{\frac{1}{2}\cdot\alpha\cdot\sigma\cdot\sigma_m\cdot(1-r-s)^{3\sigma}\sqrt{1+\frac{4s(r+s)}{\alpha\cdot\sigma_m(1-r-s)^{3\sigma}}}(1-\sqrt{1+\frac{4s(r+s)}{\alpha\cdot\sigma_m(1-r-s)^{3\sigma}}})} \in \mathbf{R},$$

because $r+s+\frac{1}{3\cdot\sigma-1}=0$ and it is an unstable saddle.
(iii) For zone III, it is an unstable saddle.
(iv) For zone IV, it is a nonlinearly equilibrium by type saddle-node.
(v) For zone V, it is is an unstable repulsive node when $\lambda_{1,2}\in\mathbf{R}_+$, and an unstable repulsive focus when $\lambda_{1,2}\notin\mathbf{R}$, with $Re\lambda_{1,2}>0.$

The Study of System 5

For System 5, we have two equilibrium points:

$$E_{2,0} = (0,0),\ and\quad E_{2,1} = (r+s, \frac{s(r+s)}{\sigma_m(1-r-s)}^{\frac{1}{1-\sigma}}).$$

The Jacobian matrix associated to System 5 is:

$$J:=\begin{pmatrix} -\sigma_m y^{1-\sigma}-s & \sigma_m(1-x)(1-\sigma)y^{-\sigma} \\ -y & r+s-x \end{pmatrix}$$

where $E=(\bar{x},\bar{y})$ is an equilibrium point.

The study of the equilibrium point $E_{2,0}$:

Since $s\in(0,1)$ we have:

Proposition 4 *The equilibrium point $E_{2,0}$ is:*

(i) an unstable saddle point for $r+s>0.$

(ii) an attractive stable node for $r+s<0.$

(iii) a nonlinearly attractive equilibrium by type saddle-node for $r+s=0$

The Characteristic Equation associated to point $E_{2,0}$ is:

$$\lambda^2 - r\lambda + (-s)(r+s) = 0,$$

with

$$\Delta_\lambda = (r+2s)^2 \Rightarrow \lambda_1 = r+s \ \text{and} \ \lambda_2 = -s,$$

because $0 < s < 1 \Rightarrow \lambda_2 < 0$ for all s.

For the equilibrium $E_{2,1}$ we obtain the following:

$$TrJ_{2,1} = -\frac{s}{(1-r-s)}$$

and

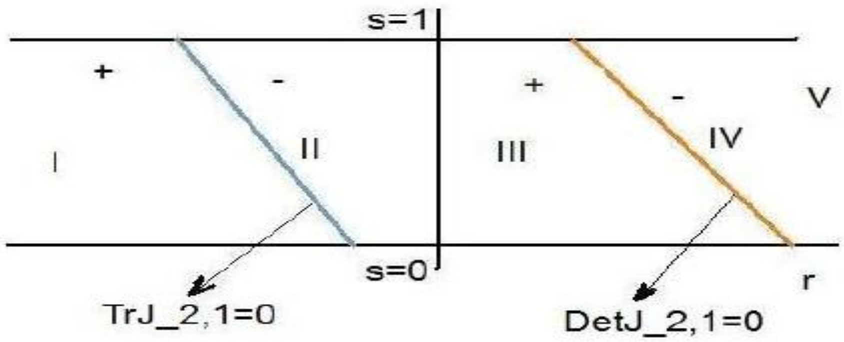
$$DetJ_{2,1} = (1-\sigma)s(r+s).$$

with its characteristic equation:

$$\lambda^2 + \frac{s}{(1-r-s)}\lambda + (1-\sigma)s(r+s) = 0.$$

Corollary 2 *There isn't Hopf bifurcation for System 5, since* $TrJ_{2,1} \neq 0$ *for any* $s \in (0,1)$.

Figure 6. Parameter portrait for the system 5 for the equilibrium point $E_{2,1}$



Source: authors' figure

Proposition 5 *The equilibrium point $E_{2,1}$ (see Figure 6) is:*

(i) For zone I, it is an unstable repulsive node when $\lambda_{1,2} \in \mathbf{R}_+$, and an unstable repulsive focus when $\lambda_{1,2} \notin \mathbf{R}$, with $Re\lambda_{1,2} > 0$.

(ii) The second zone doesn't exists, since $TrJ_{2,1} \neq 0$ for any $s \in (0,1)$.

(iii) For zone III, it is a stable attractive node when $\lambda_{1,2} \in \mathbf{R}_-$, and a stable attractive focus when $\lambda_{1,2} \notin \mathbf{R}$, with $Re\lambda_{1,2} < 0$.

(iv) For zone IV, it has the eigenvalues $\lambda_1 = 0$ and $\lambda_2 < 0$ and it is a nonlinearly attractive equilibrium by type saddle-node.

(v) For zone V, it is an unstable saddle.

V. Conclusions

The purpose of this study is to develop dynamic models in order to point out a set of facts regarding job flows and unemployment. The main results of this research are the theorems on the existence of Hopf bifurcation. The proposed models extend Ronald Shone's approach to the dynamics of unemployment by providing two different settings for the matching rate m(u,v). One of the settings is the matching function proposed by the distinguished Nobel Laureates Diamond, Mortensen and Pissarides, the other one being a personal attempt to find another function that could provide stability to a dynamic system regarding unemployment. The stability of the resulting systems is analyzed by using the software package of xpp.exe.

The most important findings are: the ode system resulting from the function that we have proposed has three equilibrium points among which the second zone of the equilibrium point E_1 is a nonlinearly point of Hopf type describing the existence of a Hopf bifurcation; the ode system resulting from using the matching function proposed by Diamond, Mortensen and Pissarides has two equilibrium points among which a nonlinearly attractive equilibrium of saddle-node type. The system resulting from using the setting proposed by Diamond, Mortensen and Pissarides is more appropriate to the economic reality, this study being another proof that their function is the most significant and important tool in analyzing the labour market dynamics. But from mathematical point of view, the system that we proposed is much more interesting because it allows the existence of Hopf bifurcation.

Acknowledgment

All our gratitude to the anonymous referees for their careful reading of the manuscript and valuable comments which helped in shaping this paper to the present form.

The author was partially supported by the strategic grant $POSDRU/CPP107/DMI1.5/\ S/78421$, Project ID $78421(2010)$, co-financed by the European Social Fund - Investing in People, within the Sectorial Operational Programme Human Resources Development $2007-2013$.

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