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Algebraic Approach to Prove Non-Coplanarity of K_9

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Abstract: *It was proved by several mathematicians in the midst of last century, almost simultaneously, that the complete graph on nine vertices is not biplanar. A new proof of this theorem is described in this article. Here, the concept of a mapping from a bipartite subgraph on six vertices of minimal lower coplanar graph to disjoint pairs of triangle in the corresponding coplanar triangulation is utilized.*
Keywords: *Coplanar Graph, Triangulation, Minimal Lower Coplanar, K_n, $K_{m,n}$*

I. BACKGROUND

In question of designing printed circuits, it was observed by John L. Selfridge that, for any graph G with p≥11 points, either G or its complement $\bar{G}$ is nonplanar. Harary [5] improved the observation for p ≥ 9 and denoted this problem as a conjecture of Selfridge in his note in 1962. In the same year Joseph Battle, Frank Harary and Yukihiro Kodama [1] gave an outline of a proof through six propositions. John R. Ball of the Carnegie Institute of Technology and W. T. Tutte [8] of the University of Waterloo prepared two contemporary proofs independently. Practically, Tutte constructed every triangulation of the sphere having 9 vertices and verified for each, that its complement is nonplanar. But Harary was not satisfied with any of these proofs. He wrote in his book [6], "This result was proved by exhaustion; no elegant or even reasonable proof is known".

D. Cvetkovic et al.[3] had a detailed study of coplanar graphs in 1991. With the help of expert system "Graph" (software), they observed the existence of 2976 coplanar graphs. In 1997, L. W. Beineke [2] had a theoretical survey on this issue.

An elegant proof for the non-coplanarity of K_9 is described in present article. Here, the concept of a one-one function from bipartite subsets of six edges on six vertices from minimal lower coplanar to a couple of disjoint triangles of the upper coplanar complement is exploited. The approach of present article seems a new concept of visualizing coplanar complements. For noncoplanar graphs, this approach has efficiency to calculate the maximum number of edges, that can be accommodated in biplane.

II. TERMINOLOGY AND NOTATION

The *complete graph* K_n has every pair of its n points adjacent. So, K_n has $\binom{n}{2}$ lines and it is *regular* (each vertex is of same degree) of degree n−1. K_1 and K_2 represent an isolated vertex and an edge respectively. In an alternative approach, a graph can be defined as an algebraic complex consisting of K_1 and K_2.

The planarity needs the help of *faces* (regions, having no subdivision), and K_3 represents a *triangle*, which is the face with smallest boundary. A *complete bipartite graph* is denoted by $K_{m,n}$, where each vertex of a set having m vertices is connected to all the n vertices of another disjoint set. It is a subclass of the triangle free graph. In a graph G, p(G) denotes the number of points or vertices, q(G) the number of lines or edges and k(G) the number of components.

The *complement* $\bar{G}$ of G, where p(G) = n, is the graph obtained by removing all the lines of G from K_n. A graph is *planar*, if it can be embedded in a plane. A planar graph is *maximal planar*, if no more edge can be added without the violation of planarity. It possess 3n−6 edges. A maximal planar graph is also called *triangulation*, since its faces are triangles. In this article a maximal planar graph on n points has been denoted by $\ddot{K}_n$ [7]. It is noteworthy that, only $\ddot{K}_5$ has unique representation. But for n ≥ 6, $\ddot{K}_n$ denotes no particular graph. Rather, it represents a class of graphs. By $\ddot{K}_n$ any particular graph of that class is referred in this article.

A planar graph, possessing a planar complement, is a *coplanar graph* [3] or *biplanar graph*. Obviously, complement of a coplanar graph is also coplanar. A lower coplanar graph is a *minimal lower coplanar graph* (L) if its complement is a maximal planar graph. Consequently, the corresponding upper complement is a *maximal upper coplanar graph* (M). It has both the properties namely, the coplanar property and the maximal planar property. A *walk* is an alternating sequence of points and lines. If the points of a walk are distinct then it is a *path*. A closed path (coincident beginning and ending point) is a *cycle* for n ≥ 3, and is denoted by C_n.

For n ≥ 4 the *wheel* W_n is defined to be the graph $K_1 + C_{n-1}$. The *double wheel* is the graph obtained from the joining of a cycle C_n with two vertices u, v by adding all possible edges from {u, v} to V (C_n), where V(C_n) is the vertex set of C_n. The set {u, v} is called the center of the double wheel and C_n is called the ring of the double wheel. Mathematically, we can denote it as D_n, where $D_n = 2K_1 + C_{n-2}$ for some $n \geq 5$ [4]. It should be noted that, double wheel is a maximal planar graph.

III. THEORETICAL DISCUSSION

Without any loss of generality, we assume that all graphs being considered are simple. A minimal lower coplanar graph has $\binom{n}{2}-(3n-6) = \frac{(n-3)(n-4)}{2} = \binom{n-3}{2}$ edges. So, a minimal lower coplanar graph, of order n, is equivalent with a complete graph of order $n-3$, in respect of the number of edges. This result is quite surprising!

In K_4, any face shares an edge with another face. But this is not true for $\ddot{K}_n$ with $n \geq 5$, because of the incompleteness of these graphs. In $\ddot{K}_5$, any face shares at least one vertex with another face. But, this feature is also unavailable in $\ddot{K}_n$ for $n \geq 6$. The following lemmas illustrate these properties on the ground of coplanarity.

Lemma 3.1. *In any triangulation on a plane for* $n \geq 4$, *to each pair of vertices we can find at least a pair of triangles, both of which include either of the vertices from the pair.*

Proof. In a planar triangulation M, suppose the pair of vertices v_i and v_j are enclosed by smallest cycles C_i and C_j (consisting of only the neighbours of the corresponding vertices) respectively. Now, any one of the following alternative cases will arise.

1. **Vertices are adjacent**: The edge v_iv_j is shared by a pair of triangles, which are enclosed by intersecting cycles C_i and C_j . But M being a triangulation with $n \geq 4$, it has no vertex of degree 2 or 1. So, C_i includes at least three triangles. Hence, C_i encloses at least one additional triangle, which includes v_i but does not include v_j .
2. **Vertices are non-adjacent**: Here, the pair of cycles does not intersect. So, any triangle enclosed within C_i cannot include v_j .

Both the cases, mentioned above are reversibly true, when the subscripts i and j are interchanged. Hence the lemma.

The following lemma describes the relation between a pair of edge disjoint triangle (may have one common vertex) of a triangulation with edges of its complement.

Lemma 3.2. *Any adjacent pair of vertices of a minimal lower coplanar graph (*L*) cannot remain on the boundary of a common face of its complement (*M*).*

Proof. If possible, let the lemma be not true, then there exists an edge x in L, whose end points are on the boundary of a face in M. But, L being the complement of M, the end vertices of x are nonadjacent in M. So, by introducing x, through the common face in M, the number of edges and faces can be improved, without any violation of planarity. This contradicts M to be a maximal planar graph.

Above lemma is very useful to characterize biplanarity.

Lemma 3.3. *In any triangulation* M *on a plane for* $n \geq 6$, *to each edge of* L *we can find at least a pair of vertex disjoint triangles or simply disjoint triangles, both of which include either of the endpoints of the edge.*

Proof. From lemmas 3.1 and 3.2, to each edge in L, the existence of a pair of edge disjoint triangles in M, having either of the endpoints, is ensured. Here, we can improve the result to obtain a disjoint pair of triangles. Suppose, V_{ij} is the set of vertices of $C_i \cap C_j$. For $V_{ij} = \emptyset$, the proof is obvious. But, for $V_{ij} \neq \emptyset$, the following alternatives arise.

1. **When** $C_i = C_j$: Then M is a double wheel with centre $\{v_i, v_j\}$. So, the common cycle has at least four vertices. A pair of adjacent vertices from the common cycle and v_i form such a triangle that has no common vertex with a triangle formed by any other pair of adjacent vertices from the common cycle and v_j.
2. **When** $C_i \neq C_j$: None of C_i or C_j can enclose the other one. So, each of them possess distinct vertices (say u_i and u_j), that are not in V_{ij} but adjacent to at least one vertex of V_{ij}. Now, following subcases arise.

(a) **If V_{ij} is a singleton set**, then by virtue of the triangulation of M, each of the cycles C_i and C_j possess at least two vertices, which are not in V_{ij}. Then, v_i along with two adjacent vertices from $C_i - V_{ij}$ and an analogous triplet of vertices related to v_j will serve our purpose.

(b) **If V_{ij} is larger than a singleton set**, either of the following subcases arises.

(i) When C_i possess only vertices of $V_{ij} \cup \{u_i\}$, then v_i, u_i and one adjacent vertex of u_i in V_{ij} will form such a triangle, that has no common vertex with the triangle formed by v_j, u_j and one adjacent vertex of u_j in V_{ij} (other than the neighbour of u_i in V_{ij}, already utilized).

(ii) When C_i possess a vertex other than those of $V_{ij} \cup \{u_i\}$, then that vertex is a better substitute of the vertices from V_{ij}.

Hence the lemma.

However, in a coplanar triangulation M, any pair of disjoint triangles is connected by edges. Otherwise, $K_{3,3}$ will be a subgraph of L, contradicting its planarity. So, the number of vertices cannot increase indiscriminately in a biplanar graph. Following lemmas improve this observation.

Lemma 3.4. *To each pair of disjoint triangles of* M *(for* $n \geq 6$*), there exist at least three edges in* L *and they form a bipartite subgraph, where the set of vertices of each triangle of the pair constitute the bi-partition.*

Proof. As M includes the pair of triangles, the rest 9 edges on six vertices (of the pair of triangles) form $K_{3,3}$, which is bipartite. But $K_{3,3}$ being nonplanar, its edges are distributed between L and M, for the sake of biplanarity. If there be less than three edges in L, the planarity of M will be contradicted. Because, by Euler's formula there exist no planar graph having six vertices and $15 - 2 = 13$ edges. Hence the lemma.

As a consequence of this lemma, there cannot have more than 6 edges in M to connect a pair of disjoint triangles. As the number of vertices increases, this number decreases. However, with the help of last lemma itself we can deduce, that this number cannot be less than 3.

Lemma 3.5. *To each pair of disjoint triangles of a coplanar triangulation* M*, there exist at least three edges in* M*, such that each of them joins a vertex of one triangle with a vertex of the other triangle.*

Proof. On the contrary, suppose in some upper coplanar triangulation M there exists a pair of disjoint triangles, which are connected by only a pair of edges. Suppose, this pair of edges constitute the subgraph $L^- (\subset K3,3)$.

Clearly, there are 7 edges in L to connect the vertices of the pair of triangles. We construct the induced subgraph of L, say M′, having those 7 edges. And then, we introduce 6 edges of the pair of triangles in M′ to get M+. So, M+ and L− form a conjugate pair in K_6. But, M+ is nonplanar by lemma 3.4. So, M′ is suspected to be nonplanar.

On the contrary, suppose M′ is planar but M+ is nonplanar. Then we delete a particular edge (that is creating nonplanarity) from M+, out of the edges of the pair of disjoint triangles of M. Next, we include the deleted edge in L−. Only this process can lead to a pair of coplanar conjugates, keeping M′ intact. But, then the lower coplanar is not bipartite relative to the bipartition formed by the vertices of each triangle of the pair from M. So, lemma 3.4 is contradicted again. Thus M′ cannot be planar, which challenges the planarity of L. Hence the lemma is established through contradiction.

Above lemma can be alternatively stated: to each pair of disjoint triangles of coplanar triangulation M, there can exist at most six edges in L.

IV. THE NON-BIPLANARITY OF K_9.

To prove the non-coplanarity of K_9 in an algebraic way, in addition to above lemmas, we need Euler's formula $p - q + r = 2$ (r is the number of faces) for polyhedra.

Theorem 4.1. *If* G *is a graph on nine points, then* G *or its complement* $\bar{G}$ *is nonplanar.*

Proof. If possible, let the theorem be not true, then there should exist a maximal upper coplanar graph $M(\subset K_9)$ and its lower complement L. Now, M has $3(9 - 2) = 21$ edges by maximal planarity, where L has $\binom{9}{2} - 21 = 15$ edges.

Next, from Euler's formula, M possess 14 faces. So, we have $\binom{14}{2}$ = 91 pairs of triangles in M. Here, we need an account of the pairs of disjoint triangle. As M has 21 edges, there are 21 pairs of triangle, which share a common edge. By virtue of triangulation, the *degree* d_i of vertex v_i ($d_i = d(v_i)$) in M is same with the number of triangles containing vertex v_i. The number of pairs of triangle, which share only vertex v_i but do not share any edge connected to v_i, is given by

$$\frac{1}{2}d_i(d_i - 3)$$

Factor $(d_i - 3)$ is due to exemption of a triangle itself and its two adjacent (edge sharing) neighbours. So, the total number of pairs of triangle in M, which share only a common vertex but not a common edge, is given by

$$D = \sum_i \frac{1}{2} d_i (d_i - 3) = \sum_i \frac{1}{2} (d_i^2 - 3d_i)$$

We cannot get an exact value of D, since it is graph dependent. Rather, we can estimate its minimum, using the theories of Inequality. It can be deduced that, D has a minimum value when the deviation of d_i's are minimum from their mean value. So, the value of D is minimum, when $\Delta(M)$ =5 for 6 vertices and $\delta(M)$ = 4 for 3 vertices ($\Delta(M)$ and $\delta(M)$ are the *maximum* and *minimum* degree of vertices in M). Thus, min D = $\frac{1}{2}[6(5^2 - 3.5) + 3(4^2 - 3.4)] = 36$. Hence, M cannot have more than 91−21−36 = 34 pairs of disjoint triangle. By lemmas 3.2 and 3.3, the edges of L follow an upper bound due to this 34 pairs of disjoint triangles. Next, we need to find out a rule to associate the edges of L with these 34 pairs of disjoint triangle of M.

Lemma 3.3 ensures that, to each edge of a minimal lower coplanar L we can find at least a pair of disjoint triangles in the maximal upper co-planar M. But it is neither one-one, nor onto relation. From mathematical point of view, such a relation for edges of L with triangles of M is a very weak relation. However, we can deduce a strong result through this relation itself, which is related with an algebraic proof of present theorem.

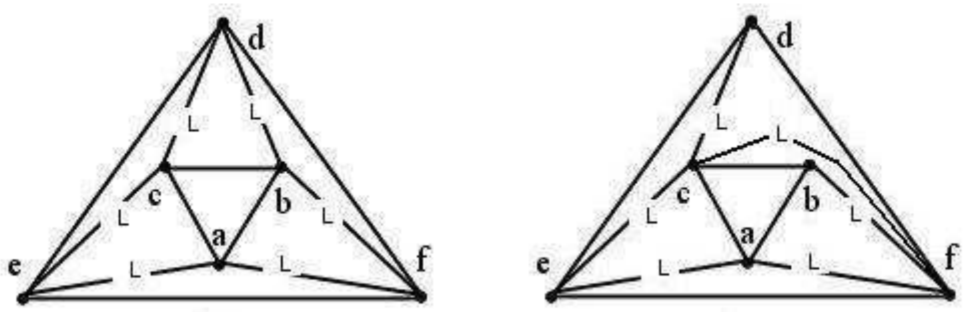

Figure 1: Examples of the a max of 6 possible edges of L, which can uniquely associate the pair of disjoint triangles Δabc and Δdef of M

By lemmas 3.4 and 3.5, to each pair of disjoint triangles of M, there exist a minimum of 3 and a maximum of 6 edges in L on 6 vertices to uniquely associate the pair of triangles. And these edges form such an induced subgraph of L that is bipartite. In order to accommodate maximum number of edges in L, corresponding to each pair of disjoint triangles of M, the associated bipartite subgraph of L should have 3 pairs of edge. We need to associate the pairs of disjoint triangle of M with the triplets of pair of associated edges of L, that form a bipartite subgraph of L. Figure 1 describes two possible examples of such cases.

As L has 15 edges, these can constitute $\binom{15}{2}$ = 105 pairs of edge in L. For each pair of disjoint triangles of M, we can find a unique collection of three pairs of edges from these 105 pairs of edges of L. The 34 pairs of disjoint triangle of M can be associated with a maximum of 102 pairs out of 105 pairs of edge. So, for at least 3 pairs of edge of L, there exists no corresponding couple of triangles in M, which contradicts lemmas 3.2 and 3.3. Hence, the non-coplanarity of K_9 is established, by the method of contradiction.

Corollary 4.1.1. *Graphs* G *and* $\bar{G}$ *can form a coplanar pair with respect to* $K_9 - x$

Proof. In $K_9 - x$, total number of edges is 35. Relative to a maximal planar graph M, having 21 edges, we may consider a lower coplanar graph L (in $K_9 - x$), that has 14 edges, which can constitute only 91 pairs of edges in L. By virtue of lemmas 3.2 and 3.3, at least $\lceil 91/3 \rceil$ = 31 pairs of disjoint triangle should exist in M, which is feasible here. Hence, the algebra permits the existence of lower coplanar graphs (having 14 edges) in $K_9 - x$ system.

In case of regular graphs, we get some advantage. For example, Icosahedron is a regular planar graph having 12 vertices, each is of degree 5.

Corollary 4.1.2. ***If we think of a lower coplanar graph like*** $\mathbf{K_9}$ **□x *system for icosahedron, the maximal possible number of edges in* L *is* 25**

Proof. For 12 vertices of icosahedron, we get 30 edges and 20 triangles in M. Next, D = $\frac{12}{2}(5^2 - 3.5) = 60$. So, we have $\binom{20}{2} - 30 - 60 = 100$ pairs of disjoint triangles, and consequently the edges of L can constitute at most 300 pairs of edges. As $\binom{25}{2} = 300$, with the help of above lemmas we find a maximum of 25 edges in L. As K_{12} has a totality of 66 edges, 11 edges cannot be accommodated in biplane.

4.1. Orthogonal relationship for edges of Lower Coplanar and faces of Upper Coplanar

An induced subgraph of L, that have six edges on six vertices, determines the existence of an unique pair of disjoint triangles in M. On the basis of this relation, in my opinion, a vector space can be designed, where a transformation from a subset of $E^6{}_L$ to a subset of $R^2{}_M$ exists. The existence of elements in E_L ensure their non existence in E_M, where R_M consists of the triangles of M and E_L contains the edges of L. Such a relation is analogous to the orthogonal relationship between a vector, that is normal to a plane, with any vector along the plane, in an Euclidean space.

V. SCOPE OF FURTHER DEVELOPMENT

The way of the proof suggests the existence of a simple algebra to estimate the maximum possible number of edges in biplane, for any complete graph. Present article suggests to define the maximal planar graph as an algebraic complex consisting of K_1, K_2 and K_3 as only constituents. Consequently, an arbitrary planar graph may be considered as a sub-complex of a maximal planar graph, where the polygons can be treated as coagulated triangles. So, a bipartite graph is suggested to be defined as a sub-complex, consisting of 2-coagulated triangles. If we like to study complete bigraphs through a similar approach, we may try with bipartite graphs on eight vertices instead of six vertices, to associate couples of quadrilaterals of the maximal upper coplanar (not triangulated, but fragmented into quadrilaterals) with the edges of lower coplanar.

This paper may also help the engineers to design three dimensional circuits more efficiently, that economise space (usually required in air-craft, submarine, etc.). There, coplanar graphs may be on the roof and the floor, along with some major devices in between them.

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