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Dominating Clique and Dominator Chromatic Number of a Prime Square Dominating Graph

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Abstract:- The aim of this paper is to put forth some revelations regarding the domination parameters of prime square dominating graphs. Some special classes of prime square dominating graphs are chosen for this purpose and their dominating cliques and dominator chromatic numbers are determined.

Keywords*:* *Prime square dominating graphs, dominating clique, independent set, domination number, dominator chromatic number*

I. INTRODUCTION

Let G be a graph with vertex set V (G) and edge set E (G). Unless specified, throughout the paper, the graph G(V, E) is a finite and connected graph. For graph theoretical terms, we refer to Harary [2] and for terms related to domination we refer to Haynes et al. [9].

Domination and the graph labeling are some of the important areas of research in graph theory. Labeling of a graph plays a vital role in graph theory. Seod M.A.and Youssef M.Z [8] and Joseph A. Gallian [5] deal with graph labeling besides many others. A labeling of a graph G is an assignment of distinct positive integers to its vertices. A graph is a prime square dominating graph, if the vertices of graph G are labeled with positive integers such that the vertex labeled with composite number c is adjacent to the vertex named with prime number p if and only if p^2 divides c.

1. 1 Definition: A subset S of the vertex set V is a dominating set of the graph if every vertex not in S is adjacent to at least one vertex of the set S. The domination number of the graph G, denoted by γ (G), is the minimum number of vertices present in a dominating set.

Dominating sets have a wide range of applications in various fields, including communication theory and combinatorics

1. 2 Definition: A set of vertices is said to be an independent set if no two vertices of the set are adjacent.

1. 3 Definition: A graph coloring is a mapping f from set of vertices V (G) to the set of colors, say C. The coloring is a proper coloring if no two adjacent vertices are assigned with the same color. A k-coloring of G is a proper coloring of G that uses at most k colors.

1. 4 Definition: The chromatic number of G, χ(G), is the minimum number of colors required to properly color the graph. Dominator coloring of a graph is a proper coloring of a graph in which each vertex of the graph dominates every vertex of some color class. The dominator chromatic number of the graph G, χ_d(G), is the minimum number of colors needed for a dominator coloring of a graph.

Dominating Set and Coloring have a number of applications and this has led to the algorithmic study of numerous variants of these problems. A number of basic combinatorial and algorithmic results on DC have been obtained and a systematic study of dominator coloring problem from the perspective of algorithms and complexity
was initiated.

1. 5 Definition: A clique dominating set or a dominating clique is a dominating set that induces a complete subgraph. A clique dominated graph is a graph that contains a dominating clique.

Wolk [1] presents a forbidden subgraph characterization of a class of graphs which have a dominating clique of size one. The concept of clique domination in graphs is introduced by Cozzens and keheller [6]. Study of Clique Domination in P_5 – free graphs is carried out by Basco [3] and bounds for clique domination number are obtained.

II. CLIQUE DOMINATION IN PRIME SQUARE DOMINATING GRAPHS

2.1 THEOREM: Let the vertex set V = { $v_1, v_2, v_3,..., v_n$}, n ≥ 2 of a prime square dominating graph G be such that a vertex v_i, 1≤ i ≤ n is labeled with a prime number p_1 and the remaining all vertices are labeled with composite numbers $c_2, c_3,... , c_n$ such that p_1^2 divides all the vertices labeled with composite numbers. Then G is a prime square dominating graph having a dominating clique of size one

Proof: Let the vertex set V={ $v_1, v_2, v_3,…, v_n$ } of the prime square dominating graph G be labeled with composite numbers and prime numbers satisfying the conditions mentioned in the theorem. Implies, vertex p_1 is adjacent to c_k ∀ k = 2, 3, 4, …, n. As there is no other vertex labeled with prime, except p_1, c_k is not adjacent to any other vertex, except p_1 for k = 2, 3, 4, ……, n. The singleton set S = { p_1 } is a dominating set and moreover it is a minimum dominating set. The subgraph induced by a single vertex is always complete. It follows that the induced subgraph < S > is a complete graph. The prime square dominating graph G has a dominating clique of size one.

2. 1. 1 Experimental problem: Consider the prime square dominating graph G with vertex set V ={3, 9, 18, 45, 63, }. It will be as shown in Fig. 1

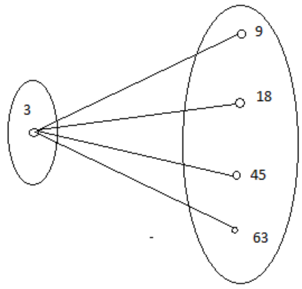

Fig. 1. Prime Square Dominating Graph

The graph satisfies all the conditions stated in the above theorem for p_1 = 3, $c_2 = 9$, $c_3 = 18$, $c_4 = 45$, $c_5 = 63$ and the order of the graph, n = 5. Clearly, minimum dominating set is S = {3}. The subgraph induced by S is also complete. Thus, S is the dominating clique of size one.

2. 2 THEOREM: Let the vertex set V = { $v_1, v_2, v_3,..., v_n$}, n ≥ 4 of a prime square dominating graph G be such that the vertices v_i and v_j, 1≤ i, j ≤ n and i≠j are labeled with a prime number p_i and composite number c_j respectively and let some of the remaining vertices be labeled with composite numbers and some with prime numbers so that p_i^2 divides all the composite numbers and all the squares of primes divide only c_j. Then G has dominating clique of size two.

Proof: Let the vertex set V = { $v_1, v_2, v_3,………, v_n$}, of a prime square dominating graph G be labeled with composite numbers and prime numbers satisfying the conditions mentioned in the theorem.

By hypothesis, except p_i, the squares of all the other primes divide only one composite number c_j. So, all the vertices labeled with primes, except p_i, are adjacent only to c_j. Moreover, all the composite numbers, except c_j, are divisible only by p_i^2. Vertex labeled with p_i is adjacent to all the vertices labeled with composite numbers and vertex labeled with c_j is adjacent to all the vertices labeled with prime numbers. The set of vertices S = {p_i ,c_j } is a dominating set. Implies, γ(G) ≤ 2. Furthermore, neither p_i nor c_j can dominate the entire set V - {p_i ,c_j }. It follows that, γ (G) >1. It is obvious that γ (G) = 2 and the set {p_i ,c_j } is the minimum dominating set. Moreover, p_i is adjacent to c_j. The induced subgraph < S > is complete and the set S is the dominating clique with minimum cardinality two.

2. 2. 1 Experimental problem: Consider the prime square dominating graph with vertex set V = {2, 3, 5, 8, 900, 12, 16}. Prime square dominating graph in this case will be as shown in Fig. 2

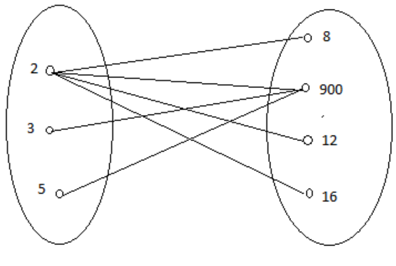

Fig. 2. Prime Square Dominating Graph

Here 2^2 divides 8, 900, 12, 16 and 3^2 divides only 900. Also, 5^2 divide only 900. The vertices p_i = 2 and c_j = 900 satisfy all the conditions mentioned in the theorem. Hence, γ (G) = 2 and the minimum dominating clique is S = { 2, 900 }. The subgraph induced by the set S will be as shown in Fig. 3

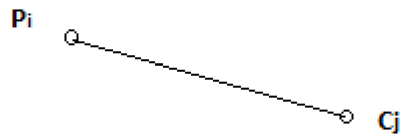

Fig. 3. The induced subgraph, < S >

2. 3 THEOREM: Let vertices of the vertex set V = { $v_1, v_2, v_3,..., v_n$}, n ≥ 4 of a prime square dominating graph G be labeled with prime numbers and composite number such that the square of every prime divides all composite numbers. Then G has dominating clique of size two and every subset of the vertex set V of order two consisting of a vertex labeled with prime and another vertex labeled with composite is a minimal dominating clique of the graph.

Proof: Let the vertex set V = { $v_1, v_2, v_3,\ldots\ldots\ldots, v_n$}, of a prime square dominating graph G be labeled with composite numbers and prime numbers satisfying the conditions mentioned in the theorem. Partition the vertex set V into two disjoint subsets V_1 and V_2, where V_1 is the set of all vertices labeled with prime numbers $p_1, p_2, \ldots, p_l$ and V_2 is the set of all vertices labeled with composite numbers $c_1, c_2, \ldots, c_m$, where l + m = n and l, m ≥ 2.

The square of every prime number divides every composite number. Every vertex labeled with prime dominates all vertices labeled with composite numbers. Every subset of the vertex set, consisting of an element from V_1 and another element from V_2 is a dominating set. Implies, γ(G) ≤ 2. Neither a vertex labeled with prime nor a vertex labeled with composite number alone can dominate all the remaining vertices. Implies, γ (G) > 1. It follows that, γ (G) = 2.

Let S_{ij} = { p_i, c_j }, where 1 ≤ i ≤ l and 1 ≤ j ≤ m. Since l, m ≥ 2 , S_{ij} is a dominating set for each possible value of i and j. Every vertex in V - S_{ij} is either labeled with a prime number or a composite number and hence will be dominated by a vertex labeled with a composite number c_j or by a vertex labeled with a prime number p_i accordingly. It follows that, the set S_{ij} is a dominating set for each possible value of i and j and furthermore, p_i dominates c_j for all permissible values of i and j. Implies, the induced subgraph < S_{ij} > is a compete graph. The sets { S_{ij} }, where 1 ≤ i ≤ l and 1 ≤ j ≤ m are all dominating cliques. Furthermore, γ (G) = 2. Hence, every set { S_{ij} },where 1 ≤ i ≤ l and 1 ≤ j ≤ m is a smallest size dominating clique.

2. 3. 1 Experimental problem: Consider the prime square dominating graph with vertex set V = {2, 3, 5, 900, 1800, 2700}. Prime square dominating graph in this case will be as shown in Fig. 4. Here 2^2 , 3^2 and 5^2 divide 900, 1800 and 2700 i.e., square of every prime divides every composite number. Hence, the sets {2, 900}, {2,1800}, {2, 2700}, {3, 900}, {3,1800}, {3, 2700}, {5, 900}, {5, 1800} and {5, 2700} are all minimum dominating cliques.

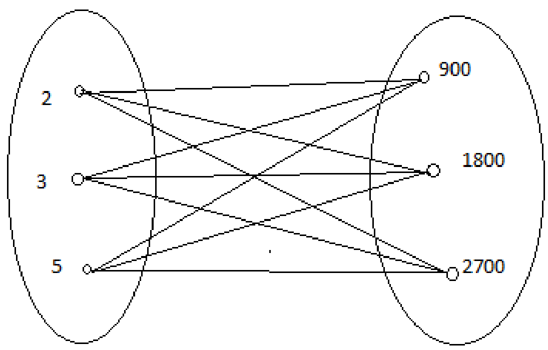

Fig. 4 Prime Square Dominating Graph

III. DOMINATOR CHROMATIC NUMBER OF SOME SPECIAL CLASSES OF PRIME SQUARE DOMINATING GRAPHS

3. 1 Some Important Results:
1.For a connected graph G of order n, 2 ≤ χ_d(G) ≤ n.
2. For any graph G, χ (G) ≤ χ_d(G) [4].
3. Max {χ (G), γ (G) } ≤ χ_d(G) ≤ χ (G) + γ(G) [7].
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3. 2 Theorem: Let the vertex set V = { v_1, v_2, v_3,..., v_n}, n ≥ 2 of a prime square dominating graph G be such that a vertex v_i, 1≤ i ≤ n is labeled with a prime number p_1 and the remaining all vertices are labeled with composite numbers c_2, c_3,... , c_n such that p_1^2 divides all the vertices labeled with composite numbers. Then the dominator chromatic number of the prime square dominating graph G is two.
Proof: It follows from theorem 3. 1 that the singleton set S = { p_1 } is a dominating set and moreover, it is a minimum dominating set. Implies, γ (G) = 1. The set of vertices labeled with composite numbers c_2, c_3,... , c_n is an independent set of vertices and can be colored with a single color. The vertex labeled with prime being adjacent to all the remaining vertices must be colored with a second different color. In this case, the prime square dominating graph G can be properly colored with two colors. It follows that, χ (G) ≤ 2. But for any connected graph, χ (G) ≥ 2. Therefore, χ (G) = 2 with two color classes C_1 = { p_1 } and C_2 = { c_2, c_3,... , c_n }. Every vertex in C_2 dominates the color class C_1 and p_1 dominates the entire color classC_2. Thus, every vertex dominates an entire class. Therefore, χ_d(G) ≤ 2. But for any graph G, χ_d(G) ≥ χ (G) = 2. Accordingly, χ_d(G) = 2.3. 2. 1 Experimental problem**:** Consider a prime square dominating graph as in the case of experimental problem 2. 1. 1. The graph satisfies all the conditions stated in the above theorem for p_1 = 3, c_2 = 9, $c_3 = 18$, $c_4 = 45$, $c_5 = 63$. The set of vertices {9, 18, 45, 63 } is an independent set and can be colored with a single color. Vertex labeled with 3 is adjacent to the vertices labeled with 9, 18, 45 and 63. So, 3 must be colored with a different color. Cearly, every vertex dominates an entire color class and furthermore, the graph G is a connected graph. Accordingly, χ_d(G) = 2 with two color classes {3} and {9, 18, 45, 63}.

3.3 THEOREM: Let the vertex set V = { v_1, v_2, v_3,..., v_n}, n ≥ 4 of a prime square dominating graph G be such that the vertices v_i and $v_{j,}$ 1≤ i, j ≤ n and i≠j are labeled with a prime number p_i and a composite number c_j respectively and let some of the remaining vertices be labeled with composite numbers and some with prime numbers so that p_i^2 divides all the composite numbers and all the squares of primes divide only c_j. Then the dominator chromatic number of the prime square dominating graph G is three.
Proof: From theorem 2. 2, it is obvious that γ (G) = 2 and the set {p_i , c_j } is the minimum dominating set. Color the vertices labeled with p_i and c_j with colors C_1' and C_2' respectively. If possible, color one of the remaining vertices with C_1'. Two cases arise.
Case i: Vertex colored with C_1' may be vertex labeled with prime, say $p_{l,}$ where l ≠ i, j and 1≤ l ≤ n. All the squares of primes divide only c_j. By the definition of prime square dominating graph, it is clear that vertex labeled with p_l does not dominate p_i but dominates c_j. As there exists another vertex that dominates p_i, let that vertex be the vertex labeled with c_m where m ≠ i, j, and 1≤ m ≤ n. The vertex c_m can be colored with C_2'. Therefore there exists a vertex labeled with prime number p_l that does not dominate the entire color class C_2'. Thus, the vertex p_l neither dominates the color class C_1' nor the color class C_2'. Another color is needed to color the vertices. As a result, χ_d(G) ≥ 3.
Case i: Vertex colored with C_1' may be vertex labeled with composite, say $c_{l,}$ where l ≠ i, j and 1≤ l ≤ n. All the squares of primes divide only c_j. By the definition of prime square dominating graph, it is clear that vertex labeled with c_l does not dominate c_j but dominates p_i. As there exists another vertex that dominates c_j, let that vertex be the vertex labeled with p_m where m ≠ i, j, and 1≤ m ≤ n. The vertex p_m can be colored with C_1'. Therefore there exists a vertex labeled with composite number c_l that does not dominate the entire color class C_1'. Thus, the vertex c_l neither dominates the color class C_1' nor the color class C_2'. Another color is needed to color the vertices. As a result, χ_d(G) ≥ 3. In both the cases, χ_d(G) ≥ 3.
Except the vertices labeled with p_i and c_j, color the remaining vertices with the color C_3'. Then, every vertex dominates an entire color class. The coloring is a dominator coloring.Thus, χ_d(G) ≤ 3. Finally, it follows that χ_d (G) = 3.3. 3. 1 Experimental problem: Consider a prime square dominating graph as in the case of experimental problem 2. 2. 1. The vertices p_i = 2 and c_j = 900 satisfy all the conditions mentioned in the theorem. Hence, γ (G) = 2. Color the vertex labeled with prime number 2 with the color C_1' and the vertex labeled with composite number 900 with the color C_2'. Color all the remaining vertices with the color C_3'.

Cearly, every vertex dominates an entire color class. In this case, the dominator chromatic number of the prime square dominating graph G is three with three color classes $C_1^{'}$ = {2}, $C_2^{'}$ = {900} and $C_3^{'}$ = {3, 5, 8, 12, 16}.

3. 4 THEOREM: Let vertices of the vertex set V = { v_1, v_2, v_3,..., v_n}, n ≥ 4 of a prime square dominating graph G be labeled with prime numbers and composite numbers such that the square of every prime divides every composite number. Then the dominator chromatic number of the prime square dominating graph G is two.
Proof: Proceeding as in the case of theorem 3. 3, it follows that γ (G) = 2 and every set S_{ij} = { p_i, q_j }, where 1 ≤ i ≤ l and 1 ≤ j ≤ m is a dominating set for each possible value of i and j. The sets of vertices { $p_1, p_2, \dots, p_l$ } and { c_1, c_2 , …, c_m } are two independent sets of vertices. All the vertices can be properly colored with two colors. As G is a connected graph, minimum two colors are needed to properly color a graph. Implies, χ (G) = 2. Let the set of all vertices { $p_1, p_2, \dots, p_l$ } be colored with $C_1^{'}$ and the set of all vertices { c_1, c_2 , …, c_m } be colored with $C_2^{'}$. By hypothesis, the square of every prime divides every composite number. Every vertex labeled with prime is adjacent to every vertex labeled with composite number. Every vertex in color class $C_1^{'}$ dominates an entire class $C_2^{'}$ and every vertex in color class $C_2^{'}$ dominates an entire class$C_1^{'}$. Every vertex dominates an entire color class. Therefore,

$$\chi_d(\mathrm{G}) \leq 2.$$

But for any graph G, χ_d(G) ≥ χ (G) = 2. Accordingly, χ_d(G) = 2.
3. 4. 1 Experimental problem: Consider a prime square dominating graph as in the case of experimental problem 2. 3. 1. Square of every prime divides every composite number. Hence, the sets {2, 900}, {2, 1800}, {2, 2700}, {3, 900}, {3, 1800}, {3, 2700}, {5, 900}, {5, 1800} and {2, 2700} are all minimal dominating sets. The sets {2, 3, 5} and {900, 1800, 2700} are independent sets and can be colored with two different colors $C_1^{'}$ and $C_2^{'}$. Every vertex in color class $C_1^{'}$ dominates every vertex in color class $C_2^{'}$. Thus, every vertex dominates an entire color class. In this case, the dominator chromatic number of the prime square dominating graph G is two with two color classes $C_1^{'}$ = {2, 3, 5}and $C_2^{'}$ = {900,1800, 2700}.

3. 5 THEOREM: If the dominator chromatic number of a prime square dominating graph is two, then every vertex labeled with prime number is adjacent to every vertex labeled with composite number.
Proof: Let χ_d(G) = 2. Among the two color classes, one color class contains vertices all labeled with primes and another color class contains vertices all labeled with composite numbers. There exists two color classes such that every vertex in one color class dominates the other color class entirely. Thus, every vertex labeled with prime number is adjacent to every vertex labeled with composite number.

IV. CONCLUSION

The findings associated to clique domination and dominator chromatic number of some special class of prime square dominating graphs are presented. In this paper, an approach to identify clique 2-domination in prime square dominating graphs is presented. This may result in opening up an avenue for many enthusiastic researchers in the field of prime square dominating graphs. In particular, in future, emphasis can be given on characterizing prime square dominating graphs with the property χ_d(G) = $\chi_d(\bar{G})$, χ_d(G) = χ (G), χ_d(G) = γ (G) and χ_d(G) = χ (G) = γ (G).

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