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Techniques to solve Non-homogeneous Ternary Sextic Diophantine Equation

$$3(x^2+y^2)-5xy=(k^2+11s^2)z^6$$

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Abstract: *New and fascinating patterns of integer solutions to non-homogeneous ternary sextic diophantine equation given by* $3(x^2+y^2)-5xy=(k^2+11s^2)z^6$ *is obtained through employing substitution technique and factorization method.*

Keywords: *Ternary sextic equation, Non-homogeneous sextic equation, Integer solutions, Substitution technique, Method of factorization*

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I. Introduction

It is well-known that a diophantine equation is an algebraic equation with integer coefficients involving two or more unknowns such that the only solutions focused are integer solutions .No doubt that diophantine equations are rich in variety [1-4] .There is no universal method available to know whether a diophantine equation has a solution or finding all solutions if it exists .For equations with more than three variables and degree at least three, very little is known. It seems that much work has not been done in solving higher degree diophantine equations. While focusing the attention on solving sextic Diophantine equations with variables at least three ,the problems illustrated in [5-24] are observed. This paper focuses on finding integer solutions to the non-homogeneous sextic equation with three unknowns $3(x^2+y^2)-5xy=(k^2+11s^2)z^6$ .In [24] ,a few sets of integer solutions to the above equation are obtained. In this paper, we present many more patterns of integer solutions for the above equation which are new and different from [24].

Method of analysis

The non-homogeneous polynomial equation of degree six with three unknowns to be solved for the integer solutions is

$$3(x^2+y^2)-5xy=(k^2+11s^2)z^6, k\neq s \qquad (1)$$

The introduction of the transformations

$$x=u+v, y=u-v, u\neq v \qquad (2)$$

in (1) leads to the sextic equation

$$u^2+11v^2=(k^2+11s^2)z^6 \qquad (3)$$

The process of obtaining varieties of non-zero integer solutions to (1) is illustrated below:

Procedure 1

By inspection ,it is seen that (3) is satisfied by

$$u = kz^3 , v = sz^3$$

In view of (2) , the values of x, y satisfying (1) are obtained as

$$x = (k+s)z^3 , y = (k-s)z^3, k \neq s$$

Procedure 2

By scrutiny, it is observed that (3) is satisfied by

$$u = k(k^2 + 11s^2)^{3\alpha} \beta^{3t} , v = s(k^2 + 11s^2)^{3\alpha} \beta^{3t} ; \beta > 1, \alpha, t > 0$$

and

$$z = (k^2 + 11s^2)^{\alpha} \beta^t \qquad (4)$$

In view of (2) ,we have

$$x = (k+s)(k^2 + 11s^2)^{3\alpha} \beta^{3t} , y = (k-s)(k^2 + 11s^2)^{3\alpha} \beta^{3t} . \qquad (5)$$

Thus , (4) & (5) satisfy (1).

Procedure 3

Rewrite (3) as

$$u^2 + 11v^2 = (k^2 + 11s^2) z^6 * 1 \qquad (6)$$

Let

$$z = a^2 + 11b^2 = (a + i\sqrt{11}b)(a - i\sqrt{11}b) \qquad (7)$$

To express $(k^2 + 11s^2)$ as the product of complex conjugates , we consider

$$k^2 + 11s^2 = (k + i\sqrt{11}s)(k - i\sqrt{11}s) \qquad (8)$$

The integer 1 on the R.H.S. of (6) is written as

$$1 = \frac{(2n^2 - 2n - 5 + i(2n-1)\sqrt{11})(2n^2 - 2n - 5 - i(2n-1)\sqrt{11})}{(2n^2 - 2n + 6)^2} \qquad (9)$$

Substituting (7) ,(8) & (9) in (6) and employing factorization , we consider

$$u+i\sqrt{11}\,v=(k+i\sqrt{11}\,s)(a+i\sqrt{11}\,b)^6\,\frac{(2n^2-2n-5+i(2n-1)\sqrt{11})}{(2n^2-2n+6)}$$
$$=(k+i\sqrt{11}\,s)[f(a,b)+i\sqrt{11}\,g(a,b)]\frac{(2n^2-2n-5+i(2n-1)\sqrt{11})}{(2n^2-2n+6)} \qquad (10)$$
$$=[f(a,b)+i\sqrt{11}\,g(a,b)]\frac{(F(k,s,n)+i\,\sqrt{11}\,G(k,s,n))}{(2n^2-2n+6)}$$

where

$$f(a,b)=a^6-165a^4b^2+1815a^2b^4-1331b^6,$$
$$g(a,b)=6a^5b-220a^3b^3+726ab^5,$$
$$F(k,s,n)=k(2n^2-2n-5)-11s(2n-1),$$
$$G(k,s,n)=s(2n^2-2n-5)+k(2n-1). \qquad (*)$$

Equating the coefficients of corresponding terms in (10) , one obtains

$$u=\frac{[f(a,b)F(k,s,n)-11g(a,b)G(k,s,n)]}{(2n^2-2n+6)},$$
$$v=\frac{[g(a,b)F(k,s,n)+f(a,b)G(k,s,n)]}{(2n^2-2n+6)}. \qquad (11)$$

As the main thrust of this paper is to obtain integer solutions , replacing a by

$(2n^2-2n+6)A$ and b by $(2n^2-2n+6)B$ in (7) & (11) and employing (2) , the

corresponding integer solutions to (1) are given by

$$x=x(A,B,k,s,n)$$
$$=(2n^2-2n+6)^5[F(k,s,n)\{f(A,B)+g(A,B)\}+G(k,s,n)\{f(A,B)-11g(A,B)\}],$$
$$y=y(A,B,k,s,n)$$
$$=(2n^2-2n+6)^5[F(k,s,n)\{f(A,B)-g(A,B)\}-G(k,s,n)\{f(A,B)+11g(A,B)\}],$$
$$z=z(A,B,n)$$
$$=(2n^2-2n+6)^2(A^2+11B^2).$$

(12)

Note 1

It is to be seen that ,in addition to (9) , one may have

$$1=\frac{(11r^2-t^2+i\sqrt{11}(2rt))(11r^2-t^2-i\sqrt{11}(2rt))}{(11r^2+t^2)^2},$$

$$1=\frac{(r^2-11t^2+i\sqrt{11}(2rt))(r^2-11t^2-i\sqrt{11}(2rt))}{(r^2+11t^2)^2}.$$

$$1=\frac{(-2n^2+2n+5+i(2n-1)\sqrt{11})(-2n^2+2n+5-i(2n-1)\sqrt{11})}{(2n^2-2n+6)^2}$$

Following the above analysis , three more sets of integer solutions to (1) are obtained.

It is noteworthy that ,the results presented in [24] are obtained by choosing the values of k,s,n in the above solutions suitably.

It is worth to mention that the expression (k^2+11s^2) is a perfect square for the following choices of k,s:

Choice 1: $k=5s$

Choice 2: $s=2t+1, k=22t^2+22t+5$

For simplicity and brevity , we present the integer solutions to (1) for Choice 1.

Procedure 4

Note that

$$k=5s\Rightarrow(k^2+11s^2)=(6s)^2$$

The substitution of the transformations

$$x=6s(u+v), y=6s(u-v), u\neq v \tag{13}$$

in (1) gives

$$u^2+11v^2=z^6 \tag{14}$$

Substituting (7) in (14) and employing factorization ,we get

$$u=f(a,b), v=g(a,b)$$

given in (*). In view of (13) ,we get

$$\begin{aligned} x&=6s[f(a,b)+g(a,b)],\\ y&=6s[f(a,b)-g(a,b)]. \end{aligned} \tag{15}$$

Thus ,(1) is satisfied by (7) & (15).

Note 2

The option

$$v=z^2 \tag{16}$$

in (14) leads to

$$u^2 = z^4(z^2 - 11)$$

which is satisfied by

$$z = \pm 6, u = \pm 180$$

and from (16) ,we obtain

$$v = 36$$

Thus , from (13) ,the following two sets of integer solutions to (1) are obtained:

$$(6^4 s, 4*6^3 s, \pm 6), (-4*6^3 s, -6^4 s, \pm 6)$$

Note 3

The choice

$$u = z^2 \tag{17}$$

in (14) gives

$$11v^2 = z^4(z^2 - 1) \tag{18}$$

Assume

$$z^2 = 11R^2 + 1 \tag{19}$$

The above equation (19) is well-known pellian equation whose general solution (z_n, R_n) is given by

$$z_n = \frac{f_n}{2}, R_n = \frac{g_n}{2\sqrt{11}} \tag{20}$$

where

$$\begin{aligned} f_n &= (10+3\sqrt{11})^{n+1} + (10-3\sqrt{11})^{n+1}, \\ g_n &= (10+3\sqrt{11})^{n+1} - (10-3\sqrt{11})^{n+1}. \end{aligned} \tag{21}$$

From (18) ,we get

$$v_n = z_n^2 * R_n$$

From (17) ,we have

$$u_n = z_n^2$$

In view of (13) , we obtain

$$x_n = 6s(1+R_n)z_n^{\,2} = \frac{3s}{4\sqrt{11}}[2\sqrt{11}+g_n]f_n^{\,2},$$
$$y_n = 6s(1-R_n)z_n^{\,2} = \frac{3s}{4\sqrt{11}}[2\sqrt{11}-g_n]f_n^{\,2}. \qquad (22)$$

Thus , the values of x_n, y_n, z_n given by (22) & (20) satisfy (1).

In a similar manner ,one may determine the integer solutions to (1) for Choice 2.

II. Conclusion

In this paper ,an attempt has been made to determine the non-zero distinct integer solutions to the non-homogeneous sextic diophantine equation with three unknowns given in the title through employing transformations. The researchers in this area may search for other choices of transformations to obtain integer solutions to the sextic diophantine equation with three or more unknowns.

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