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A Transformed Optimum Estimator under Incomplete Information in Sample Survey

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Summary
A transformed ratio-product-cum-difference estimator for estimating the universe mean $\bar{Y}$ of the principal variable y in presence of non-response has been envisaged. The properties of the recommended family have been studied. The merits of the recommended family of estimators with other competitors are also worked out. To show the performance of the recommended family of estimators over other competitors an appropriate example is given.
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I. Introduction

It has been observed in practice that generally almost all surveys be effected from the problem of non-response (*NR).* The lack of information, absence at the time of survey, and refusal of the respondent are the main causes of *NR*. Hansen and Hurwitz (1946) investigated a simple procedure of sub-sampling the non-respondents in order to adopt for the *NR* in a mail surveys. For estimating the universe mean $\bar{Y}$ of the main variable y with increased precision the use of information available on subsidiary variable x can be done. If the universe mean $\bar{X}$ is known and in presence of *NR*, the problem of estimating $\bar{Y}$ has been furnished due to Cochran (1977), Rao (1986), Khare and Srivastava (1997), Okafor and Lee (2000), Kumar et al. (2011), Olufadi and Kumar (2014), Chanu and Singh (2015) and Pal and Singh (2016) etc.

Suppose a finite universe $U=(U_1,U_2,...,U_N)$ of N units. Let (y,x) be the (principal, subsidiary) variates respectively taking value (y_i,x_i) on units $U_i (i=1,2,...,N)$. In human population surveys, frequently n_1 units respond on the items under examination at first trial while remaining $n_2(=n-n_1)$ units do not supply any answer. If *NR* happened in the initial trial, Hansen and Hurwitz (1946) reported a double sampling scheme for estimating $\bar{Y}$ of y comprising the following steps: (i) a simple random sample (*SRS*) of size n is selected and the questionnaire is mailed to the sample units, (ii) a sub-sample of size $r=n_2 k^{-1},(k\geq 1)$; from the n_2 non-responding units in the initial attempt is contacted through personal interviews.

Thus universe is assumed to be composed of two strata of sizes N_1 and $N_2=(N-N_1)$ respondents and non-respondents. Thus $\bar{Y}$ can be written as $\bar{Y}=D_1\bar{Y}_1+D_2\bar{Y}_2$, where $D_1=(N_1/N)$, $D_2=(N_2/N)$ and ($\bar{Y}_1=N_1^{-1}\sum_{i=1}^{N_1}y_i$, $\bar{Y}_2=N_2^{-1}\sum_{i=1}^{N_2}y_i$) are means of 'responding' and 'non-responding' groups in the universe. Let $\bar{y}_1=n_1^{-1}\sum_{i=1}^{n_1}y_i$ and $\bar{y}_2=n_2^{-1}\sum_{i=1}^{n_2}y_i$ denote the means of n_1 responding units and n_2 non-responding units respectively and $\bar{y}_{2r}=r^{-1}\sum_{i=1}^{r}y_i$.

Hansen and Hurwitz (1946) recommended an unbiased estimator for $\bar{Y}$ of y as

$$\bar{y}^*=d_1\bar{y}_1+d_2\bar{y}_{2r}, \qquad (1.1)$$

where $d_1=n_1/n$ and $d_2=n_2/n$.

The variance of $\bar{y}^*$ is given by

$$V(\bar{y}^*) = \frac{(1-f)}{n} S_y^2 + \frac{D_2(k-1)}{n} S_{y(2)}^2 = \bar{Y}^2 V_y, \tag{1.2}$$

where $V_y = \left\{ \frac{(1-f)}{n} C_y^2 + \frac{D_2(k-1)}{n} C_{y(2)}^2 \right\}$, $f = \frac{n}{N}$, S_y^2 and $S_{y(2)}^2$ are mean squares / variances for the whole universe and for the *NR* group of the universe respectively.

Rao (1986) suggested the ratio estimator as

$$\bar{y}_R^* = \bar{y}^* \frac{\bar{X}}{\bar{x}^*}. \tag{1.3}$$

The product estimator in the presence of *NR* formulated as

$$\bar{y}_P^* = \bar{y}^* \frac{\bar{x}^*}{\bar{X}}, \tag{1.4}$$

where $\bar{x}^* = d_1 \bar{x}_1 + d_2 \bar{x}_{2r}$ with $\bar{x}_1$ and $\bar{x}_{2r}$ being the sample means based on n_1 and r observations on x respectively.

Using the power transformation the generalized version of the estimators $\bar{y}^*$, $\bar{y}_R^*$ and $\bar{y}_P^*$ respectively defined in Eq. (1.1), (1.2) and Eq. (1.3) is given by

$$\bar{y}_{SR} = \bar{y}^* \left(\frac{\bar{X}}{\bar{x}^*} \right)^{\alpha}, \tag{1.5}$$

where α is a suitable chosen scalars, for instance, see Srivastava(1967).

If we set $\alpha = 2$ in Eq. (1.5) we get an estimator analogous to the Kadilar and Cingi's (2003) chain-type estimator for $\bar{Y}$ in presence of *NR* as

$$\bar{y}_{KC} = \bar{y}^* \left(\frac{\bar{X}}{\bar{x}^*} \right)^2. \tag{1.6}$$

Putting $\alpha = 1/2$ in Eq. (1.5) we get another estimator analogous to the Swain's (2014) estimator in presence of *NR* for $\bar{Y}$ as

$$\bar{y}_{SW} = \bar{y}^* \left(\frac{\bar{X}}{\bar{x}^*} \right)^{1/2}. \tag{1.7}$$

With known $\bar{X}$ and incomplete data on y and x, the estimator for $\bar{Y}$ due to Khare and Srivastava (1997) is given by

$$\bar{y}_{KS} = \bar{y}^* \left(\frac{\bar{X} + b}{\bar{x}^* + b} \right), \tag{1.8}$$

where b being a constant.

When the complete information on the both characters y and x are available and $\bar{X}$ of x is known, Srivenkataramana (1980) and Bandyopadhyaya (1980) suggested a dual to ratio estimator for estimating $\bar{Y}$ as

$$\bar{y}_{C1} = \bar{y} \frac{\{(1+g)\bar{X} - g\bar{x}\}}{\bar{X}}, \tag{1.9}$$

where $g = [n/(N-n)]$.

Let $(\bar{y}, \bar{x})$ designated as the *SRS* means of (y, x) respectively based on complete information of the sample of size n.

Replacing $(\bar{y}, \bar{x})$ by $(\bar{y}^*, \bar{x}^*)$ respectively in Eq. (1.9) we get an estimator for $\bar{Y}$ in presence of *NR* on both the variables (y, x) as

$$\bar{y}_{SB} = \bar{y}^* \frac{\{(1+g)\bar{X} - g\bar{x}^*\}}{\bar{X}}. \qquad (1.10)$$

Further, when the complete information on (y, x) is available, Srivenkataramana and Tracy (1981) have suggested an alternative estimator for $\bar{Y}$ as

$$\bar{y}_{C2} = \bar{y}\left(\frac{b - \bar{x}}{b - \bar{X}}\right), \qquad (1.11)$$

where b being a suitably chosen scalar.

Replacing $(\bar{y}, \bar{x})$ by $(\bar{y}^*, \bar{x}^*)$ respectively in Eq. (1.11), we derive an estimator for the $\bar{Y}$ in presence of *NR* on (y, x) as

$$\bar{y}_{ST} = \bar{y}^*\left(\frac{b - \bar{x}^*}{b - \bar{X}}\right), \qquad (1.12)$$

where b being a constant.

Keeping the structure of the above estimators in view we have recommended a family of *RPCD* estimators for $\bar{Y}$ of y in presence of *NR*. We have obtained bias and *MSE* of the recommended family up to order n^{-1}. Condition is obtained for which the *MSE* of the recommended family is minimized. An appropriate example is given.

II. The recommended Family of Estimators

We suggest the family of ratio-product-cum-difference (*RPCD*) estimators for $\bar{Y}$ as

$$t = \bar{y}^*\left(\frac{a\bar{X} + b}{a\bar{x}^* + b}\right)^{\alpha} + d(\bar{X} - \bar{x}^*), \qquad (2.1)$$

where (α, d, a, b) are the real numbers or the parameters associated with principal variable y or suitable variable x or both variable (y, x). We note that for $(\alpha, d) = (1,0)$ and suitable choices of (a, b), one can define a large number of estimators [see Sisodia and Dwivedi (1981), Sahai and Sahai (1985), Upadhyaya and Singh (1999) and Kadilar and Cingi (2006) etc].

Now, we have

$\bar{y}^* = \bar{Y}(1 + e_0)$, $\bar{x}^* = \bar{X}(1 + e_1)$ such that

$E(e_0) = E(e_1) = 0$ and $E(e_0^2) = V_y$, $E(e_1^2) = V_x$, $E(e_0 e_1) = V_{xy}$,

where

$$V_x = \left\{\frac{(1-f)}{n} C_x^2 + \frac{D_2(k-1)}{n} C_{x(2)}^2\right\}, V_{xy} = \left\{\frac{(1-f)}{n} \rho C_y C_x + \frac{D_2(k-1)}{n} \rho_{(2)} C_{y(2)} C_{x(2)}\right\},$$

$$C_y = \frac{S_y}{\bar{Y}}, C_x = \frac{S_x}{\bar{X}},\ \rho = \frac{S_{xy}}{S_x S_y}, C_{y(2)} = \frac{S_{y(2)}}{\bar{Y}}, C_{x(2)} = \frac{S_{x(2)}}{\bar{X}},\ \rho_{(2)} = \frac{S_{xy(2)}}{S_{x(2)} S_{y(2)}},$$

$$S_y^2 = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2, S_{y(2)}^2 = (N_2 - 1)^{-1}\sum_{i=1}^{N_2}(y_i - \bar{Y}_2)^2,$$

$$S_x^2 = (N-1)^{-1}\sum_{i=1}^{N}(x_i - \bar{X})^2, S_{x(2)}^2 = (N_2 - 1)^{-1}\sum_{i=1}^{N_2}(x_i - \bar{X}_2)^2,$$

$$S_{xy} = (N-1)^{-1}\sum_{i=1}^{N}(x_i - \bar{X})(y_i - \bar{Y}), S_{xy(2)} = (N-1)^{-1}\sum_{i=1}^{N2}(x_i - \bar{X}_2)(y_i - \bar{Y}_2),$$

$(S_y^2, S_x^2, S_{xy}, \rho)$ being the variances of (y, x), covariances, respectively for the whole universe, ($S_{y(2)}^2, S_{x(2)}^2, S_{xy(2)}, \rho_{(2)}$) being the variances of (y, x), covariances, respectively for '*NRP*' group in the universes.

We express Eq. (2.1) as

$$t = \bar{Y}[(1+e_0)\{1+\tau e_1\}^{-\alpha}] - d\bar{X}e_1, \tag{2.2}$$

where $\tau = [a\bar{X}/(a\bar{X}+b)]$.

The expressions (2.2) approximated as

$$t \cong \bar{Y}\left[1 + e_0 - \alpha\tau e_1 + \alpha\tau e_0 e_1 + \frac{\alpha(\alpha+1)}{2}\tau^2 e_1^2\right] - d\bar{X}e_1$$

or

$$(t - \bar{Y}) \cong \bar{Y}\left[e_0 - \left(\alpha\tau + \frac{d}{R}\right)e_1 + \alpha\tau e_0 e_1 + \frac{\alpha(\alpha+1)}{2}\tau^2 e_1^2\right]. \tag{2.3}$$

where $R = \bar{Y}/\bar{X}$.

Taking expectation of the above expression we get the bias of recommended family 't':

$$B(t) = \bar{Y}\alpha\tau\left[\frac{(1-f)}{n}C_x^2\left\{\frac{(\alpha+1)}{2}\tau - C\right\} + \frac{(1-f)D_2}{n}C_{x(2)}^2\left\{\frac{(\alpha+1)}{2}\tau - C_{(2)}\right\}\right], \tag{2.4}$$

where $C = \rho\frac{C_y}{C_x}$ and $C_{(2)} = \rho_{(2)}\frac{C_{y(2)}}{C_{x(2)}}$.

When the 'n' is sufficiently large the bias of 't'at Eq. (2.4) is negligible. Also, the bias of t at Eq. (2.4) would be zero if either $\alpha = 0$ or $\alpha = \left(\frac{2C}{\tau} - 1\right)$ and $\alpha = \left(\frac{2C_{(2)}}{\tau} - 1\right)$.

Squaring both sides of Eq. (2.3), we write the approximated expression as

$$(t - \bar{Y})^2 = \bar{Y}^2\left[e_0^2 + \left(\alpha\tau + \frac{d}{R}\right)^2 e_1^2 - 2\left(\alpha\tau + \frac{d}{R}\right)e_0 e_1\right]. \tag{2.5}$$

Thus *MSE* of 't' is provided as

$$MSE(t) = \bar{Y}^2\left[V_y + V_x\left(\alpha\tau + \frac{d}{R}\right)\left\{\alpha\tau + \frac{d}{R} - 2R_c\right\}\right] \tag{2.6}$$

which is minimized for

$$\left(\alpha\tau + \frac{d}{R}\right) = R_c \tag{2.7}$$

Thus the minimum mean squared error (*MMSE*) of t is given by

$$MMSE(t) = \bar{Y}^2[V_y - V_x R_c^2] \tag{2.8}$$

we thus established theorem given below.

Theorem 2.1: Up to order n^{-1}

$$MSE(t) \geq \bar{Y}^2[V_y - V_x R_c^2]; \text{ if } \left(\alpha\tau + \frac{d}{R}\right) = R_c.$$

We write the approximate *MSE*s of the estimators $\bar{y}_R^*, \bar{y}_P^*, \bar{y}_{SR}, \bar{y}_{KC}, \bar{y}_{SW}, \bar{y}_{KS}, \bar{y}_{SB}$ and $\bar{y}_{ST}$ are respectively, as

$$MSE(\bar{y}_R^*) = \bar{Y}^2[V_y + V_x(1 - 2R_c)], \tag{2.9}$$

$$MSE(\bar{y}_P^*) = \bar{Y}^2[V_y + V_x(1 + 2R_c)], \tag{2.10}$$

$$MSE(\bar{y}_{SR}) = \bar{Y}^2[V_y + \alpha V_x(\alpha - 2R_c)], \tag{2.11}$$

$$MSE(\bar{y}_{KC}) = \bar{Y}^2[V_y + 4V_x(1 - R_c)], \tag{2.12}$$

$$MSE(\bar{y}_{SW}) = \bar{Y}^2[V_y + (V_x/4)(1 - 4R_c)], \tag{2.13}$$

$$MSE(\bar{y}_{KS}) = \bar{Y}^2[V_y + V_x\tau_1(\tau_1 - 2R_c)], \quad (2.14)$$

$$MSE(\bar{y}_{SB}) = \bar{Y}^2[V_y + V_x g(g - 2R_c)], \quad (2.15)$$

$$MSE(\bar{y}_{ST}) = \bar{Y}^2[V_y + V_x\tau_2(\tau_2 - 2R_c)], \quad (2.16)$$

where $\tau_1 = \dfrac{\bar{X}}{\bar{X}+b}$ and $\tau_2 = \dfrac{\bar{X}}{\bar{X}-b}$.

From Eq. (1.2), (2.8) and Eq. (2.9) to (2.16) we have

$$Var(\bar{y}^*) - MMSE(t) = \bar{Y}^2 V_x R_c^2 \geq 0, \quad (2.17)$$

$$MSE(\bar{y}_R^*) - MMSE(t) = \bar{Y}^2 V_x (1 - R_c)^2 \geq 0, \quad (2.18)$$

$$MSE(\bar{y}_P^*) - MMSE(t) = \bar{Y}^2 V_x (1 + R_c)^2 \geq 0, \quad (2.19)$$

$$MSE(\bar{y}_{SR}) - MMSE(t) = \bar{Y}^2 V_x (\alpha - R_c)^2 \geq 0, \quad (2.20)$$

$$MSE(\bar{y}_{KC}) - MMSE(t) = \bar{Y}^2 V_x (2 - R_c)^2 \geq 0, \quad (2.21)$$

$$MSE(\bar{y}_{SW}) - MMSE(t) = \bar{Y}^2 (V_x/4)(1 - 2R_c)^2 \geq 0, \quad (2.22)$$

$$MSE(\bar{y}_{KS}) - MMSE(t) = \bar{Y}^2 V_x (\tau_1 - R_c)^2 \geq 0, \quad (2.23)$$

$$MSE(\bar{y}_{SB}) - MMSE(t) = \bar{Y}^2 V_x (g - R_c)^2 \geq 0, \quad (2.24)$$

$$MSE(\bar{y}_{ST}) - MMSE(t) = \bar{Y}^2 V_x (\tau_2 - R_c)^2 \geq 0. \quad (2.25)$$

From Eq. (2.17) to (2.25) we note that the recommended family of *RPCD* estimator is more accurate than $\bar{y}^*$, $\bar{y}_R^*$, $\bar{y}_P^*$, $\bar{y}_{SR}$, $\bar{y}_{KC}$, $\bar{y}_{SW}$, $\bar{y}_{KS}$, $\bar{y}_{SB}$ and $\bar{y}_{ST}$.

Further, we note that the *MSE*s of $\bar{y}_{SR}$, $\bar{y}_{KS}$ and $\bar{y}_{ST}$ are minimized for

$$\alpha = \tau_1 = \tau_2 = R_c. \quad (2.26)$$

Thus the resulting common *MMSE* of $\bar{y}_{SR}$, $\bar{y}_{KS}$ and $\bar{y}_{ST}$ is given by

$$MMSE(\bar{y}_{SR}) = MMSE(\bar{y}_{KS}) = MMSE(\bar{y}_{ST}) = \bar{Y}^2[V_y - V_x R_c^2] \quad (2.27)$$

Thus it is observed from Eq. (2.8) and Eq. (2.27) that the estimators 't', $\bar{y}_{SR}$, $\bar{y}_{KS}$ and $\bar{y}_{ST}$ are equally efficient at optimum condition.

III. Efficiency Comparison of the Recommended Family 't' When the Optimum Value Does Not Coincide With its True Value

3.1 Comparability of the recommended family 't' with $\bar{y}^*$

From Eq. (1.2) and Eq. (2.14) we have

$$Var(\bar{y}^*) - MSE(t) = \bar{Y}^2 V_x \left(\alpha\tau + \frac{d}{R}\right)\left(2R_c - \alpha\tau - \frac{d}{R}\right)$$

which is positive if

$$\left.\begin{array}{ll} \text{either} & 0 < \left(\alpha\tau + \dfrac{d}{R}\right) < 2R_c \\ \text{or} & 2R_c < \left(\alpha\tau + \dfrac{d}{R}\right) < 0 \end{array}\right\} \quad (3.1)$$

which follows that the recommended family 't' is better than $\bar{y}^*$ as long as the Eq. (3.1) is satisfied.

3.2 Comparability of the recommended family 't' with $\bar{y}_R^*$

Now from Eq. (2.6) and Eq. (2.9) we have

$$MSE(\bar{y}_R^*) - MSE(t) = \bar{Y}^2 V_x \left[1 - 2R_c - \left(\alpha\tau + \frac{d}{R}\right)^2 + 2R_c\left(\alpha\tau + \frac{d}{R}\right)\right]$$ which is positive if

$$\left.\begin{array}{ll} \text{either} & (2R_c - 1) < \left(\alpha\tau + \frac{d}{R}\right) < 1 \\ \text{or} & 1 < \left(\alpha\tau + \frac{d}{R}\right) < (2R_c - 1) \end{array}\right\} \quad (3.2)$$

Thus the recommended family 't' is more precise than $\bar{y}_R^*$ as long as Eq. (3.2) is satisfied.

3.3 Comparability of the recommended family 't' with $\bar{y}_P^*$

We have from Eq. (2.6) and Eq. (2.10):

$$MSE(\bar{y}_P^*) - MSE(t) = \bar{Y}^2 V_x \left[1 + 2R_c - \left(\alpha\tau + \frac{d}{R}\right)^2 + 2R_c\left(\alpha\tau + \frac{d}{R}\right)\right]$$ which is positive if

$$\left.\begin{array}{ll} \text{either} & (2R_c + 1) < \left(\alpha\tau + \frac{d}{R}\right) < -1 \\ \text{or} & -1 < \left(\alpha\tau + \frac{d}{R}\right) < (2R_c + 1) \end{array}\right\} \quad (3.3)$$

Thus under the Eq. (3.3) the envisaged family 't' is superior to $\bar{y}_P^*$.

3.4 Comparability of the recommended family 't' with $\bar{y}_{SR}$

Let the constant α be preassined.

Then from Eq. (2.6) and Eq. (2.11) we have

$$MSE(\bar{y}_{SR}) - MSE(t) = \bar{Y}^2 V_x \left[\alpha^2 - 2\alpha R_c - \left(\alpha\tau + \frac{d}{R}\right)^2 + 2R_c\left(\alpha\tau + \frac{d}{R}\right)\right]$$

which is non-negative if

$$\left.\begin{array}{ll} \text{either} & (2R_c - \alpha) < \left(\alpha\tau + \frac{d}{R}\right) < \alpha \\ \text{or} & \alpha < \left(\alpha\tau + \frac{d}{R}\right) < (2R_c - \alpha) \end{array}\right\} \quad (3.4)$$

Thus the recommended family 't' is more precise than the Srivastava (1967) estimator $\bar{y}_{SR}$ if the Eq. (3.4) is satisfied.

3.5 Comparability of the recommended family 't' with $\bar{y}_{KC}$

From Eq. (2.6) and Eq. (2.12) we have

$$MSE(\bar{y}_{KC}) - MSE(t) = \bar{Y}^2 V_x \left[4 - 4R_c - \left(\alpha\tau + \frac{d}{R}\right)^2 + 2R_c\left(\alpha\tau + \frac{d}{R}\right)\right]$$

which is positive if

$$\left.\begin{array}{ll}\text{either} & 2(R_c - 1) < \left(\alpha\tau + \dfrac{d}{R}\right) < 2 \\ \text{or} & 2 < \left(\alpha\tau + \dfrac{d}{R}\right) < 2(R_c - 1)\end{array}\right\} \quad (3.5)$$

Thus the recommended family 't' is more efficient than Kadilar and Cingi's (2003) chain-type estimator $\bar{y}_{KC}$ as long as the Eq. (3.5) is satisfied.

3.6 Comparability of the recommended family 't' with $\bar{y}_{SW}$

From Eq. (2.6) and Eq. (2.13) we have

$$MSE(\bar{y}_{SW}) - MSE(t) = \bar{Y}^2 V_x \left[\frac{1}{4} - R_c - \left(\alpha\tau + \frac{d}{R}\right)^2 + 2R_c\left(\alpha\tau + \frac{d}{R}\right)\right]$$

which is positive if

$$\left.\begin{array}{ll}\text{either} & \left(2R_c - \dfrac{1}{2}\right) < \left(\alpha\tau + \dfrac{d}{R}\right) < \dfrac{1}{2} \\ \text{or} & \dfrac{1}{2} < \left(\alpha\tau + \dfrac{d}{R}\right) < \left(2R_c - \dfrac{1}{2}\right)\end{array}\right\} \quad (3.6)$$

Thus the recommended family 't' is more accurate than Swain's (2014) ratio-type estimator $\bar{y}_{SW}$ as long as the Eq. (3.17) is satisfied.

3.7 Comparability of the recommended family 't' with $\bar{y}_{KS}$

Let 'b' be the pre-assigned constant.
Then from Eq. (2.6) and Eq. (2.14) we have

$$MSE(\bar{y}_{KS}) - MSE(t) = \bar{Y}^2 V_x \left[\left(\tau_1 + \frac{d}{R}\right)^2 - 2R_c\left(\tau_1 + \frac{d}{R}\right) - \left(\alpha\tau + \frac{d}{R}\right)^2 + 2R_c\left(\alpha\tau + \frac{d}{R}\right)\right]$$

which is positive if

$$\left.\begin{array}{ll}\text{either} & \left\{2R_c - \left(\tau_1 + \dfrac{d}{R}\right)\right\} < \left(\alpha\tau + \dfrac{d}{R}\right) < \left(\tau_1 + \dfrac{d}{R}\right) \\ \text{or} & \left(\tau_1 + \dfrac{d}{R}\right) < \left(\alpha\tau + \dfrac{d}{R}\right) < \left\{2R_c - \left(\tau_1 + \dfrac{d}{R}\right)\right\}\end{array}\right\} \quad (3.7)$$

Thus the recommended family 't' is more efficient than the estimator Khare and Srivastava (1997) $\bar{y}_{KS}$ as long as the Eq. (3.20) is satisfied.

3.8 Comparability of the recommended family 't' with $\bar{y}_{SB}$

From Eq. (2.6) and Eq. (2.15) we have

$$MSE(\bar{y}_{SB}) - MSE(t) = \bar{Y}^2 V_x \left[g^2 - 2gR_c - \left(\alpha\tau + \frac{d}{R}\right)^2 + 2R_c\left(\alpha\tau + \frac{d}{R}\right)\right]$$

which is positive if

$$\left.\begin{array}{ll}\text{either} & (2R_c - g) < \left(\alpha\tau + \dfrac{d}{R}\right) < g \\ \text{or} & g < \left(\alpha\tau + \dfrac{d}{R}\right) < (2R_c - g)\end{array}\right\} \quad (3.8)$$

Thus the recommended family 't' is better than the Srivenkataramana's (1980) and Bandyopadhyaya's (1980) estimator $\bar{y}_{SB}$ if the Eq. (3.8) is holds well.

3.9 Comparability of the recommended family 't' with $\bar{y}_{ST}$

Let 'b' be a pre-assigned constant. Then from Eq. (2.6) and (2.16) we have

$$MSE(\bar{y}_{ST}) - MSE(t) = \bar{Y}^2 V_x\left[\tau_2^2 - 2R_c\tau_2 - \left(\alpha\tau + \frac{d}{R}\right)^2 + 2R_c\left(\alpha\tau + \frac{d}{R}\right)\right]$$

which is non-negative if

$$\left.\begin{array}{ll} \text{either} & \{2R_c - \tau_2\} < \left(\alpha\tau + \frac{d}{R}\right) < \tau_2 \\ \text{or} & \tau_2 < \left(\alpha\tau + \frac{d}{R}\right) < \{2R_c - \tau_2\} \end{array}\right\} \quad (3.9)$$

Thus the recommended family 't' is more precise than Srivenkataramana and Tracy's (1981)-type estimator $\bar{y}_{ST}$ if the Eq. (3.9) is satisfied.

Now, we consider three special cases discussed in the Section 4.

IV. Some Special Cases

Case I: For $(a,b,\alpha,d) = (1, C_x, 1, 1)$ in Eq. (2.1), family 't' reduce to:

$$t_{(1)} = \bar{y}^*\left(\frac{\bar{X} + C_x}{\bar{x}^* + C_x}\right) + (\bar{X} - \bar{x}^*). \quad (4.1)$$

Putting $(a,b,\alpha,d) = (1, C_x, 1, 1)$ in Eq. (2.6) we get the approximate *MSE* of $t_{(1)}$ as

$$MSE(t_{(1)}) = \bar{Y}^2\left[V_y + V_x\left(\tau_{(1)} + \frac{1}{R}\right)\left(\tau_{(1)} + \frac{1}{R} - 2R_c\right)\right], \quad (4.2)$$

where $\tau_{(1)} = \dfrac{\bar{X}}{(\bar{X} + C_x)}$.

Further if we set $(a,b,\alpha,d) = (1, C_x, 1, 0)$ in Eq. (2.1) we get the estimator for $\bar{Y}$ as

$$\bar{y}_{SD} = \bar{y}^*\left(\frac{\bar{X} + C_x}{\bar{x}^* + C_x}\right) \quad (4.3)$$

which is Sisodia and Dwivedi's (1981) ratio-type estimator.

The approximate *MSE* of $\bar{y}_{SD}$ is given by

$$MSE(\bar{y}_{SD}) = \bar{Y}^2[V_y + V_x\tau_{(1)}\{\tau_{(1)} - 2R_c\}] \quad (4.4)$$

From Eq. (1.2), (2.9), (2.12), (2.13), (2.15), (4.2) and Eq. (4.3), it is observed that $t_{(1)}$ is superior than:

(i) $\bar{y}^*$ if

$$R > \frac{(R_1^* + 1)}{2R_c} \quad (4.5)$$

where $R_1^* = [\bar{Y}/(\bar{X} + C_x)]$.

(ii) $\bar{y}_R^*$ if

$$\left.\begin{array}{lll} \text{either} & (1+R_1^*) < R < \left(\dfrac{1+R_1^*}{2R_c-1}\right), & R_c > \dfrac{1}{2} \\ \text{or} & \left(\dfrac{1+R_1^*}{2R_c-1}\right) < R < (1+R_1^*) & R_c < \dfrac{1}{2} \end{array}\right\}. \tag{4.6}$$

(iii) the Kadilar and Cingi's(2003) chain ratio-type estimator $\bar{y}_{KC}$ if

$$\left.\begin{array}{lll} \text{either} & \left(\dfrac{1+R_1^*}{2}\right) < R < \left(\dfrac{1+R_1^*}{2(R_c-1)}\right), & R_c > 1 \\ \text{or} & \left(\dfrac{1+R_1^*}{2(R_c-1)}\right) < R < \left(\dfrac{1+R_1^*}{2}\right), & R_c < 1 \end{array}\right\}. \tag{4.7}$$

(iv) the Swain's (2014) ratio-type estimator $\bar{y}_{SW}$ if

$$\left.\begin{array}{lll} \text{either} & 2\left(1+R_1^*\right) < R < \dfrac{2(1+R_1^*)}{(4R_c-1)}, & R_c > \dfrac{1}{4} \\ \text{or} & \dfrac{2(1+R_1^*)}{(4R_c-1)} < R < 2\left(1+R_1^*\right), & R_c < \dfrac{1}{4} \end{array}\right\}. \tag{4.8}$$

(v) $\bar{y}_{SB}$ if

$$\left.\begin{array}{lll} \text{either} & \left(\dfrac{1+R_1^*}{g}\right) < R < \left(\dfrac{1+R_1^*}{2R_c-g}\right), & R_c > \dfrac{g}{2} \\ \text{or} & \left(\dfrac{1+R_1^*}{2R_c-g}\right) < R < \left(\dfrac{1+R_1^*}{g}\right), & R_c < \dfrac{g}{2} \end{array}\right\}. \tag{4.9}$$

(vi) $\bar{y}_{SD}$ if

$$R > \frac{\left(1+2R_1^*\right)}{2R_c}. \tag{4.10}$$

Case II: [when the correlation coefficients ρ between y and x is known]

For $(a,b,\alpha,d)=(1,\rho,1,1)$ in Eq. (2.1), we get another estimator for $\bar{Y}$ as

$$t_{(2)} = \bar{y}^*\left\{\frac{\bar{X}+\rho}{\bar{x}^*+\rho}\right\} + (\bar{X}-\bar{x}^*). \tag{4.11}$$

The approximate *MSE* of $t_{(2)}$ is given by

$$MSE(t_{(2)}) = \bar{Y}^2\left[V_y + V_x\left(\tau_{(2)}+\frac{1}{R}\right)\left(\tau_{(2)}+\frac{1}{R}-2R_c\right)\right], \tag{4.12}$$

where $\tau_{(2)} = \dfrac{\bar{X}}{(\bar{X}+\rho)}$.

If we set $(a,b,\alpha,d)=(1,\rho,1,0)$, we get the Singh and Tailor's (2003)-type estimator for $\bar{Y}$ as

$$\bar{y}_{ST(1)} = \bar{y}^*\left(\frac{\bar{X}+\rho}{\bar{x}^*+\rho}\right). \tag{4.13}$$

The approximate MSE of $\bar{y}_{ST(1)}$, is given by

$$MSE(\bar{y}_{ST(1)}) = \bar{Y}^2[V_y + V_x \tau_{(2)}\{\tau_{(2)} - 2R_c\}]. \quad (4.14)$$

From Eq. (1.2), (2.9), (2.12), (2.13), (2.15), (4.2) and Eq. (4.12), we note that the recommended estimator $t_{(2)}$ is more efficient than:

(i) the unbiased estimator $\bar{y}^*$ if

$$R > \frac{(R_2 + 1)}{2R_c}, \quad (4.15)$$

where $R_2 = \dfrac{\bar{Y}}{(\bar{X} + \rho)}$.

(ii) the ratio estimator $\bar{y}_R^*$ if

$$\left.\begin{array}{lll} \text{either} & (1+R_2) < R < \left(\dfrac{1+R_2}{2R_c - 1}\right), & R_c > \dfrac{1}{2} \\ \text{or} & \left(\dfrac{1+R_2}{2R_c - 1}\right) < R < (1+R_2) & R_c < \dfrac{1}{2} \end{array}\right\}, \quad (4.16)$$

(iii) the Kadilar and Cingi's(2003) chain ratio-type estimator $\bar{y}_{KC}$ if

$$\left.\begin{array}{lll} \text{either} & \left(\dfrac{1+R_2}{2}\right) < R < \left(\dfrac{1+R_2}{2(R_c - 1)}\right), & R_c > 1 \\ \text{or} & \left(\dfrac{1+R_2}{2(R_c - 1)}\right) < R < \left(\dfrac{1+R_2}{2}\right), & R_c < 1 \end{array}\right\}, \quad (4.17)$$

(iv) the Swain's (2014) ratio-type estimator $\bar{y}_s$ if

$$\left.\begin{array}{lll} \text{either} & 2(1+R_2) < R < \dfrac{2(1+R_2)}{(4R_c - 1)}, & R_c > \dfrac{1}{4} \\ \text{or} & \dfrac{2(1+R_2)}{(4R_c - 1)} < R < 2(1+R_2), & R_c < \dfrac{1}{4} \end{array}\right\}, \quad (4.18)$$

(v) the dual to ratio-type estimator $\bar{y}_{SB}$ if

$$\left.\begin{array}{lll} \text{either} & \left(\dfrac{1+R_2}{g}\right) < R < \left(\dfrac{1+R_2}{2R_c - g}\right), & R_c > \dfrac{g}{2} \\ \text{or} & \left(\dfrac{1+R_2}{2R_c - g}\right) < R < \left(\dfrac{1+R_2}{g}\right), & R_c < \dfrac{g}{2} \end{array}\right\}, \quad (4.19)$$

(vi) the estimator $\bar{y}_{SD}$ if

$$R > \frac{(1 + 2R_{21})}{2R_c}. \quad (4.20)$$

Case III: We consider the following families of estimators of $\bar{Y}$ as

$$t_{(3)} = \bar{y}^* \left(\frac{\bar{X} + C_x}{\bar{x}^* + C_x}\right)^{\alpha} + (\bar{X} - \bar{x}^*) \quad (4.21)$$

$$t_{(4)} = \bar{y}^*\left(\frac{\bar{X}+\rho}{\bar{x}^*+\rho}\right)^{\alpha} + (\bar{X}-\bar{x}^*) \tag{4.22}$$

$$t_{(5)} = \bar{y}^*\left(\frac{C_x\bar{X}+\rho}{C_x\bar{x}^*+\rho}\right)^{\alpha} + (\bar{X}-\bar{x}^*) \tag{4.23}$$

$$t_{(6)} = \bar{y}^*\left(\frac{\rho\bar{X}+C_x}{\rho\bar{x}^*+C_x}\right)^{\alpha} + (\bar{X}-\bar{x}^*) \tag{4.24}$$

The approximate *MSE*s of $t_{(j)}$, $j = 3 \text{to} 6$ are given as

$$MSE(t_{(j)}) = \bar{Y}^2\left[V_y + V_x\left(\alpha\tau_{(j)} + \frac{1}{R}\right)\left\{\alpha\tau_{(j)} + \frac{1}{R} - 2R_c\right\}\right], \tag{4.25}$$

where $\tau_{(1)} = \tau_{(3)} = \dfrac{\bar{X}}{\bar{X}+C_x}$, $\tau_{(2)} = \tau_{(4)} = \dfrac{\bar{X}}{\bar{X}+\rho}$, $\tau_{(5)} = \dfrac{C_x\bar{X}}{C_x\bar{X}+\rho}$ and $\tau_{(6)} = \dfrac{\rho\bar{X}}{\rho\bar{X}+C_x}$.

To demonstrate that $t_{(j)}$, $j = 3 \text{to} 6$ is superior to:

(i) $\bar{y}^*$ if

$$\min.\left\{-\frac{1}{R\tau_{(j)}}, \frac{1}{\tau_{(j)}}\left(2R_c - \frac{1}{\text{R}}\right)\right\} < \alpha < \max.\left\{-\frac{1}{R\tau_{(j)}}, \frac{1}{\tau_{(j)}}\left(2R_c - \frac{1}{\text{R}}\right)\right\}, \tag{4.26}$$

(ii) $\bar{y}_R^*$ if

$$\min.\left\{\frac{1}{\tau_{(j)}}\left(1-\frac{1}{R}\right), \frac{1}{\tau_{(j)}}\left(-1-\frac{1}{\text{R}} - 2R_c\right)\right\} < \alpha < \max.\left\{\frac{1}{\tau_{(j)}}\left(1-\frac{1}{R}\right), \frac{1}{\tau_{(j)}}\left(-1-\frac{1}{\text{R}} - 2R_c\right)\right\}, \tag{4.27}$$

(iii) $\bar{y}_P^*$ if

$$\min.\left\{-\frac{1}{\tau_{(j)}}\left(1+\frac{1}{R}\right), \frac{1}{\tau_{(j)}}\left(1-\frac{1}{\text{R}} + 2R_c\right)\right\} < \alpha < \max.\left\{-\frac{1}{\tau_{(j)}}\left(1+\frac{1}{R}\right), \frac{1}{\tau_{(j)}}\left(1-\frac{1}{\text{R}} + 2R_c\right)\right\}. \tag{4.28}$$

The $MSE(t_{(j)})$ is minimized for

$$\alpha_{(j0)} = \frac{1}{t_{(j)}}\left(R_c - \frac{1}{R}\right). \tag{4.29}$$

Thus the resulting *MMSE* of $t_{(j)}$, $j = 3 \text{to} 6$ is given by

$$MMSE(t_{(j)}) = \bar{Y}^2[V_y - V_x R_c^2]. \tag{4.30}$$

It demonstrates that the performance of $t_{(j)}$, $j = 3 \text{to} 6$ are equally efficient with the suggested estimator 't' at their most select conditions.

V. The Recommended Family of Estimators in the Presence of *NR* only on y

Let the complete information on x be known for the sample of size n and that incomplete information on y be known. Thus, we use information on $(n_1 + r)$ responding units on the principal (main) character y, and the complete information on the subsidiary character x from the sample of size n.

We suggest a family of *RPCD* estimators for $\bar{Y}$ as

$$t^* = \bar{y}^*\left(\frac{a\bar{X}+b}{a\bar{x}+b}\right) + d(\bar{X}-\bar{x}), \tag{5.1}$$

For $(\alpha, d) = (0,0)$, t^* reduces to $\bar{y}^*$ while for $(a,b,\alpha,d) = (1,0,1,0)$, it boils down to the estimator t_R^* as

$$t_R^* = \bar{y}^* \left(\frac{\bar{X}}{\bar{x}} \right). \tag{5.2}$$

The approximate bias and *MSE* of t^* are respectively given below

$$B(t^*) = \frac{(1-f)}{n} \left(\frac{\bar{Y}\alpha\tau}{2} \right) C_x^2 [(\alpha+1)\tau - 2C], \tag{5.3}$$

$$MSE(t^*) = \bar{Y}^2 \left[V_y + \frac{(1-f)}{n} \left(\alpha\tau + \frac{d}{R} \right) \left\{ \left(\alpha\tau + \frac{d}{R} \right) - 2C \right\} \right], \tag{5.4}$$

The biases and *MSE*s of the estimators included to the family t^* given by Eq. (5.1) can be obtained from Eq. (5.3) and Eq. (5.4) for suitable values of (a,b,α,d), respectively.

The MSE of t^* at Eq. (5.4) is minimized for

$$\left(\alpha\tau + \frac{d}{R} \right) = C, \tag{5.5}$$

which gives the *MMSE* of t^* as

$$MMSE(t^*) = \bar{Y}^2 \left[\frac{(1-f)}{n} C_y^2 (1-\rho^2) + \frac{D_2(k-1)}{n} C_{y(2)}^2 \right]. \tag{5.6}$$

So we established theorem given below.

Theorem 5.1: Up to order n^{-1}

$$MSE(t^*) \geq \bar{Y}^2 \left[\frac{(1-f)}{n} C_y^2 (1-\rho^2) + \frac{D_2(k-1)}{n} C_{y(2)}^2 \right], \text{ if } \left(\alpha\tau + \frac{d}{R} \right) = C.$$

VI. Empirical Study

In this section a natural population data set earlier used by Khare and Sinha (2004, p.53) is taken. The value of parameters associated to y [the weight in Kg.] and x [the chest circumference in cm.] are given below:

$\bar{Y} = 19.4968$, $\bar{X} = 55.8611$, $S_y = 3.0435$, $S_x = 3.2735$, $S_{y(2)} = 2.3552$, $S_{x(2)} = 2.5137$,

$\rho = 0.8660$, $\rho_{(2)} = 0.7290$, $R = 0.3490$, $D_2 = 0.25$, $N = 95$, $n = 35$.

We have enumerated percentage relative efficiencies (*PRE*s) of different estimators relative to $\bar{y}^*$ with the help of the formulae given below:

$$PRE(\bar{y}_R^*, \bar{y}^*) = \frac{V_y}{[V_y + V_x(1-2R_c)]} \times 100, \tag{6.1}$$

$$PRE(\bar{y}_{SB}, \bar{y}^*) = \frac{V_y}{[V_y + V_x g(g-2R_c)]} \times 100, \tag{6.2}$$

$$PRE(\bar{y}_{KC}, \bar{y}^*) = \frac{V_y}{[V_y + V_x(1-R_c)]} \times 100, \tag{6.3}$$

$$PRE(\bar{y}_{SW}, \bar{y}^*) = \frac{V_y}{\left[V_y + \left(\frac{V_x}{4} \right)(1-4R_c) \right]} \times 100, \tag{6.4}$$

$$PRE(t_{(1)},\bar{y}^*)=\frac{V_y}{\left[V_y+V_x\left(\tau_{(1)}+\frac{1}{R}\right)\left(\tau_{(1)}+\frac{1}{R}-2R_c\right)\right]}\times 100, \quad (6.5)$$

$$PRE(\bar{y}_{SD},\bar{y}^*)=\frac{V_y}{[V_y+V_x(\tau_{(1)}-2R_c)\tau_{(1)}]}\times 100\ , \quad (6.6)$$

$$PRE(t_{(2)},\bar{y}^*)=\frac{V_y}{\left[V_y+V_x\left(\tau_{(2)}+\frac{1}{R}\right)\left(\tau_{(2)}+\frac{1}{R}-2R_c\right)\right]}\times 100, \quad (6.7)$$

$$PRE(t_{ST(1)},\bar{y}^*)=\frac{V_y}{[V_y+V_x\tau_{(2)}(\tau_{(2)}-2R_c)]}\times 100, \quad (6.8)$$

where $\tau_{(1)}=\frac{\bar{X}}{\bar{X}+C_x}$, $\tau_{(2)}=\frac{\bar{X}}{\bar{X}+\rho}$.

We further computed the *PRE* of the families of estimators $t_{(j)}$, $j=3\text{to}6$ using the formula:

$$PRE(t_{(j)},\bar{y}^*)=\frac{V_y}{\left[V_y+V_x\left(\alpha\tau_{(j)}+\frac{1}{R}\right)\left\{\alpha\tau_{(j)}+\frac{1}{R}-2R_c\right\}\right]},\ j=3\text{to}6.$$

We have also computed the range of α for $t_{(3)},t_{(4)}$, $t_{(5)}$ and $t_{(6)}$ to be more precise than $\bar{y}^*$, $\bar{y}_R^*$ and $\bar{y}_P^*$ by using the formula in Eq.(4.26) to Eq.(4.28).
Results are tabulated in Tables 6.1, 6.2 and 6.3.

Table 6.1: *PRE*s of $\bar{y}_R^*$, $\bar{y}_{SB}$, $\bar{y}_{KC}$, $\bar{y}_{SW}$, $t_{(1)}$, $\bar{y}_{SD}$, $t_{(2)}$ and $\bar{y}_{ST(1)}$ with respect to $\bar{y}^*$

$PRE(\bullet,\bar{y}^*)$				
k	$\bar{y}_R^*$	$\bar{y}_{SB}$	$\bar{y}_{KC}$	$\bar{y}_{SW}$
5	181.9835	142.2047	118.3871	135.1750
4	184.1254	142.9704	118.8259	135.7686
3	187.0347	143.9951	119.4108	136.5611
2	191.2138	145.4369	120.2295	137.6728
k	$t_{(1)}$	$\bar{y}_{SD}$	$t_{(2)}$	$\bar{y}_{ST(1)}$
5	123.8817	181.8757	124.9307	180.4515
4	127.5150	184.0128	128.6144	182.5262
3	132.6570	186.9155	133.8297	185.3425
2	140.4934	191.0849	141.7821	189.3845

In general the values of *PRE*s of different estimators ($\bar{y}_R^*$, $\bar{y}_{SB}$, $\bar{y}_{KC}$, $\bar{y}_S$, $t_{(1)}$, $\bar{y}_{SD}$, $t_{(2)}$ and $\bar{y}_{ST(1)}$) increase for decreasing value of k. In Table 6.1, the ratio estimator $\bar{y}_R^*$ appears to be the best as it has the largest *PRE* among the estimators discussed. Further we note that the performance of the estimators $\bar{y}_{SD}$ and $\bar{y}_{ST(1)}$ are almost at par with $\bar{y}_R^*$.

Table 6.2: Range of α for the $t_{(j)}$, $j=3\text{to}6$ to be better than $\bar{y}^*$, $\bar{y}_R^*$ and $\bar{y}_P^*$, optimum value of α and $PRE(t_{(j)},\bar{y}^*)$ $j=3\text{to}6$ at optimum α.

$t_{(3)}$					
k	$\bar{y}^*$	$\bar{y}_R^*$	$\bar{y}_P^*$	Optimum α	PRE at optimum α
5	(-2.8680,1.3570)	(-1.8670, 0.3560)	(-3.8690, 2.3580)	-0.7555	265.2220
4	(-2.8680,1.3991)	(-1.8670, 0.3981)	(-3.8690, 2.4001)	-0.7344	274.8381
3	(-2.8680,1.4547)	(-1.8670, 0.4537)	(-3.8690, 2.4557)	-0.7066	288.8290
2	(-2.8680,1.5313)	(-1.8670, 0.5303)	(-3.8690, 2.5323)	-0.6683	311.0505
$t_{(4)}$					
5	(-2.9085, 1.3761)	(-1.8934, 0.3610)	(-3.9236, 2.3913)	-0.7662	265.2220
4	(-2.9085, 1.4189)	(-1.8934, 0.4038)	(-3.9236, 2.4340)	-0.7448	274.8381
3	(-2.9085, 1.4752)	(-1.8934, 0.4601)	(-3.9236, 2.4904)	-0.7166	288.8290
2	(-2.9085, 1.5529)	(-1.8934, 0.5378)	(-3.9236, 2.5681)	-0.6778	311.0505
$t_{(5)}$					
5	(-3.6058, 1.7061)	(-2.3473, 0.4476)	(-4.8643, 2.9645)	-0.9499	265.2220
4	(-3.6058, 1.7591)	(-2.3473, 0.5006)	(-4.8643, 3.0176)	-0.9234	274.8381
3	(-3.6058, 1.8289)	(-2.3473, 0.5704)	(-4.8643, 3.0874)	-0.8884	288.8290
2	(-3.6058, 1.9252)	(-2.3473, 0.6668)	(-4.8643, 3.1837)	-0.8403	311.0505
$t_{(6)}$					
5	(-2.8686, 1.3573)	(-1.8674, 0.3561)	(-3.8698, 2.3585)	-0.7557	265.2220
4	(-2.8686, 1.3994)	(-1.8674, 0.3982)	(-3.8698, 2.4006)	-0.7346	274.8381
3	(-2.8686, 1.4550)	(-1.8674, 0.4538)	(-3.8698, 2.4562)	-0.7068	288.8290
2	(-2.8686, 1.5316)	(-1.8674, 0.5304)	(-3.8698, 2.5328)	-0.6685	311.0505

Table 6.2 shows that the range of α in which the recommended estimators $t_{(3)}$ to $t_{(6)}$ are superior to $\bar{y}^*$, $\bar{y}_R^*$ and $\bar{y}_P^*$ along with most favorable values and $PRE(t_{(j)}, \bar{y}^*)$ for $j = 3 \text{ to } 6$ at optimum value of α . It is observed from Table 6.2 that the length of the interval increases with decreasing value of k . That in the range of α becomes wider for decreasing value of k and the absolute optimum value of α (i.e. $|\alpha_{opt}|$) decreases with decreasing value of k . So for varying k better scope of choosing the scalar α is observed to obtain the better estimators than $\bar{y}^*$, $\bar{y}_R^*$ and $\bar{y}_P^*$.

Table 6.3: PREs of $t_{(j)}$, $j = 3 \text{ to } 6$ relative to $\bar{y}^*$ for different values of α

Estimator	k \ α	-1.8670	-1.7500	-1.5000	-1.2500	-1.0000	-0.7500
$PRE(t_{(3)}, \bar{y}^*)$	5	181.9937	194.1452	220.0708	243.2098	259.4818	265.2190
	4	184.1360	196.8664	224.3460	249.3840	267.5923	274.8127
	3	187.0459	200.5821	230.2578	258.0414	279.1226	288.6101
	2	191.2259	205.9591	238.9681	271.0575	296.8134	310.1491
	k \ α	-0.5000	-0.2500	0.0000	0.2500	0.3560	
	5	258.9614	242.2972	218.9514	192.9847	181.9838	
	4	269.1556	252.1133	227.6719	200.2897	188.6756	
	3	283.9271	266.3755	240.3218	210.8373	198.3131	
	2	307.2516	288.9892	260.3297	227.4019	213.3903	
$PRE(t_{(4)}, \bar{y}^*)$	k \ α	-1.8933	-1.7500	-1.5000	-1.2500	-1.0000	-0.7500
	5	181.9925	196.6897	222.1574	244.6101	260.1031	265.1970
	4	184.1348	199.5431	226.5803	250.9225	268.3128	274.8354
	3	187.0446	203.4438	232.7031	259.7824	279.9924	288.7026

	2	191.2246	209.0976	241.7390	273.1221	297.9315	310.3640
	α \ k	-0.5000	-0.2500	0.0000	0.2500	0.3610	
	5	258.6252	242.0089	218.9514	193.3476	181.9851	
	4	268.8215	251.8138	227.6719	200.6729	188.6770	
	3	283.5983	266.0599	240.3218	211.2505	198.3146	
	2	306.9361	288.6480	260.3297	227.8643	213.3920	
	α \ k	-2.3473	-1.7500	-1.5000	-1.2500	-1.0000	-0.7500
	5	181.9727	230.6299	247.6607	259.7393	265.0656	262.7645
	4	184.1141	235.6950	254.2874	267.8904	274.4459	272.8465
	3	187.0227	242.7406	263.6091	279.4818	287.9116	287.4214
$PRE(t_{(5)},\bar{y}^{*})$	2	191.2008	253.2105	277.6900	297.2742	308.8743	310.3533
	α \ k	-0.5000	-0.2500	0.0000	0.2500	0.4476	
	5	253.2202	237.9266	218.9514	198.3390	181.9785	
	4	263.3695	247.5643	227.6719	205.9418	188.6700	
	3	278.1007	261.5671	240.3218	216.9316	198.3071	
	2	301.4217	283.7689	260.3297	234.2207	213.3836	
	α \ k	-1.8674	-1.7500	-1.5000	-1.2500	-1.0000	-0.7500
	5	181.9892	194.1801	220.0996	243.2293	259.4905	265.2188
	4	184.1313	196.9031	224.3768	249.4053	267.6024	274.8131
	3	187.0409	200.6213	230.2915	258.0656	279.1348	288.6115
$PRE(t_{(6)},\bar{y}^{*})$	2	191.2205	206.0020	239.0062	271.0861	296.8291	310.1523
	α \ k	-0.5000	-0.2500	0.0000	0.2500	0.3561	
	5	258.9569	242.2932	218.9514	192.9897	181.9805	
	4	269.1510	252.1091	227.6719	200.2950	188.6721	
	3	283.9227	266.3712	240.3218	210.8429	198.3094	
	2	307.2474	288.9845	260.3297	227.4083	213.3862	

Table 6.3 demonstrates that the *PRE*s of the families $t_{(j)}$, $j=3\text{to}6$ relative to $\bar{y}^{*}$ for varying (α, k). It is found from Table 6.3 that for fixed α the *PRE*s of $t_{(j)}$, $j=3\text{to}6$ increase for decreasing values of k. The *PRE*s of $t_{(j)}$'s, $j=3\text{to}6$ related to $\bar{y}^{*}$ are larger when the scalar α moves in the neighborhood of its optimum value. From Table 6.2 it is seen that the largest gain in efficiencies are observed at optimum value of α which is expected too.

Further it is derived from Tables 6.1 and 6.3 that the performance of the estimator $t_{(j)}$, $j=3\text{to}6$ are appreciable as compared to the other competitors ($\bar{y}_{R}^{*}, \bar{y}_{SB}, \bar{y}_{KC}, \bar{y}_{SW}, t_{(1)}, \bar{y}_{SD}$, $t_{(2)}$ and $\bar{y}_{ST(1)}$). Finally we conclude from above discussion that there is large scope of picking the value of α to get the estimators superior than $\bar{y}^{*}$, $\bar{y}_{R}^{*}$ and $\bar{y}_{P}^{*}$ from the recommended family $t_{(j)}$, $j=3\text{to}6$.

The recommended family 't' defined by Eq. (2.1) is very wide and a large number of useful and acceptable estimators for $\bar{Y}$ of y can be generated from it for various values of scalars (a,b,α,d). So our recommendation is in the favour of recommended families $t_{(j)}$, $j=3\text{to}6$ and the estimator 't'.

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