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Simple Semirings

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Abstract: *Author determine different additive structures of simple semiring which was introduced by Golan [3]. We also proved some results based on the papers of Fitore Abdullahu [1].*

I. Introduction

This paper reveals the additive structures of simple semirings by considering that the multiplicative semigroup is rectangular band.

1.1. Definition: A semigroup S is called medial if xyzu = xzyu, for every x, y, z, u in S.

1.2. Definition: A semigroup S is called left (right) semimedial if it satisfies the identity $x^2yz = xyxz$ ($zyx^2 = zxyx$), where $x,y,z \in S$ and x, y are idempotent elements.

1.3. Definition: A semigroup S is called a semimedial if it is both left and right semimedial.

Example: The semigroup S is given in the table is I-semimedial

*	a	b	c
a	b	b	b
b	b	b	b
c	c	c	c

1.4. Definition: A semigroup S is called I- left (right) commutative if it satisfies the identity xyz = yxz (zxy = zyx), where x, y are idempotent elements.

1.5. Definition: A semigroup S is called I-commutative if it satisfies the identity xy = yx, where $x,y \in S$ and x, y are idempotent elements.

Example: The semigroup S is given in the table is I-commutative.

*	a	b	c
a	b	b	a
b	b	b	b
c	c	b	c

1.6. Definition: A semigroup S is called I-left(right) distributive if it satisfies the identity xyz = xyxz (zyx = zxyx), where $x,y,z \in S$ and x, y are idempotent elements.

1.7. Definition: A semigroup S is called I-distributive if it is both left and right distributive

1.8. Definition: A semigroup S is said to be cancellative for any a, b, $\in$S, then $ac = bc \Rightarrow a = b$ and $ca = cb \Rightarrow a = b$

1.9. Definition: A semigroup S is called diagonal if it satisfies the identities $x^2 = x$ and xyz = xz.

1.10. Definition: A regular semigroup S is said to be generalized inverse semigroup if all its idempotent elements form a normal band.

1.11. Definition: A regular semigroup S is said to be locally inverse semigroup if eSe is an inverse semigroup for any idempotent e in S.

1.12. Definition: A regular semigroup S is said to be orthodox semigroup if E(S) is subsemigroup of S.

1.14. Definition: An element "a" of S is called k-regular if a^k is regular element, for any positive integer k. If every element of S is k-regular then S is k-regular semigroup.

Examples:
(i) If S is a zero semigroup on a set with zero, then S is 2-regular but not regular, has a unique idempotent namely, zero but is not a group.
(ii) A left (right) zero semigroup is k-regular for every positive integer k but not k-inverse unless it is trivial. It is to be noted that regular = 1- regular.

1.15. Definition: [3] A semiring S is called simple if a + 1 = 1 + a = 1 for any $a \in S$.

1.17. Definition: A semiring (S, +, .) is called an additive inverse semiring if (S, +) is an inverse semigroup, i.e., for each a in S there exists a unique element $a^1 \in S$ such that $a + a^1 + a = a$ and $a^1 + a + a^1 = a^1$

Example: Consider the set S = {0, a, b} on S we define addition and multiplication by the following cayley tables then S is additive inverse semiring.

+	0	a	b
0	0	a	b
a	a	0	b
b	b	b	b

•	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

1.18Definition: A semiring S is called a regular semiring if for each $a \in S$ there exist an element $x \in S$ such that a = axa.

Examples: (i) Every regular ring is a regular semiring
(ii) Every distributive lattice is regular semiring.
(iii) The direct product of regular ring and distributive lattice is regular semiring.

1.19. Definition: An additive idempotent semiring S is k-regular if for all a in S there is x in S for which a + axa = axa.

Example:
Let D be a distributive lattice. Consider S = $M_2(D)$ the semiring of 2×2 matrices on D. Now consider $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then for. $X=\begin{bmatrix} a & c \\ b & d \end{bmatrix}$
A + AXA = AXA and this shows that S is k-regular.

1.20. Theorem: A simple semiring is additive idempotent semiring.
Proof: Let (S, +, .) be a simple semiring.Since (S, +, .) is simple, for any $a \in S$, a + 1 = 1. (Where 1 is the multiplicative identity element of S. S^1 = SU {1}.)
Now a = a.1 = a(1 + 1) = a + a $\Rightarrow$ a = a + a $\Rightarrow$ S is additive idempotent semiring.

1.21. Theorem: Every singular semigroup (S, +) is (i) semilattice
(ii) Rectangular band.

1.22. Theorem: Let (S, +, .) be a simple semiring. Then S is regular semiring if and only if it is k-regular semiring.

Proof: Let (S, +, .) be a simple semiring.Since (S, +, .) is a simple, for any $a \in S$, a + 1 =1.

Consider a = a.1= a(1 + b) = a + ab $\Rightarrow$ a = a + ab

Similarly, a = a + ba

Assume that S be a regular semiring. Then for any $a \in S$ there exist an element x in S such that axa = a. Now consider axa = a = a.1 = a(1 + x) = a + ax = a + a (x + xa) = a + ax + axa = a + axa $\Rightarrow$ axa = a + axa. Hence (S, +, .) is k-regular semiring.

Conversely, assume that (S, +, .) be a k-regular semiring then for any $a \in S$ there exist $x \in S$ such that axa = a + axa. Now consider a + axa = a(1 + xa) = a.1 = a $\Rightarrow$ axa = a $\Rightarrow$ S is a regular semiring

1.23. Theorem: Let (S, +, .) be a simple semiring. If (S, +, .) is k- regular semiring then S is additively regular semiring.

Proof: Let (S, +, .) be a simple semiring.Since (S, +, .) is simple, for any $a \in S$, a + 1 = 1. Since S be a k-regular semiring, for any $a \in S$ there exist an element x in S such that axa = a + axa. To Prove that S is additively regular semiring, consider axa = a + axa $\Rightarrow$ a = a + a (by Theorem 4.2.1.) $\Rightarrow$ (S, +) is a band

Since (S, +) is a band then clearly (S, +) is regular. Therefore S is additively regular semiring.

1.24. Theorem: If the idempotent elements of a regular semigroup are commutes then S is generalized inverse semigroup.

Proof: Let S be a regular semigroup whose idempotent elements are commutes. Let x, y, z $\in S$ and are idempotent elements then xyz = xzy $\Rightarrow$ xxyz = xxzy $\Rightarrow$ xyxz = xzxy $\Rightarrow$ xyzx = xzyx. Hence idempotent elements form a normal band. Therefore S is generalized inverse semigroup.

1.25. Theorem: Let S be a regular semigroup and E(S) is an E-inversive semigroup of S then i) E(S) is subsemigroup ii) S is an orthodox semigroup iii) E(S) is locally inverse semigroup.

Proof: (i) Let S be a regular semigroup and E(S) is an E-inversive semigroup. If $a, b \in S$ then there exist some x, y in S such that

$(ax)^2 = (ax)$ and $(by)^2 = (by)$, $(xa)^2 = (xa)$ and $(yb)^2 = (yb)$

Let $(axby)^2 = (ax)^2 (by)^2 = (ax)(by) \Rightarrow (axby)^2 = (ax)(by)$.

(axby) is an idempotent of E(S).

Hence E(S) is subsemigroup of S and its elements are idempotents.

(ii) Since E(S) is a sub semigroup of S $\Rightarrow$ S is an orthodox semigroup.

(iii) To prove that eE(S)e is regular for some $e \in S$. Let $f \in E(S)$ then

(efe)(efe)(efe) = efeefeefe = efefefe = efe $\Rightarrow$ (efe)(efe)(efe) = efe

Similarly (fef)(fef)(fef) = feffeffef = fefefef = fef $\Rightarrow$ (fef)(fef)(fef) = fef $\Rightarrow$ efe is an inverse element of E(S) $\Rightarrow$ E(S) is locally inverse semigroup.

1.26. Theorem: Let (S, +, .) be a semiring. Then the following statements are equivalent:

(i) a + 1 = 1 (ii) $a^n + 1 = 1$ (iii) $(ab)^n + 1 = 1$ For all $a,b \in S$.

Proof is by mathematical induction

1.27. Theorem: Let (s,+,.) be simple semiring. Then for any $a,b \in S$ the following holds:

(i) a+b+1 = 1 (ii) ab +1 = 1.

1.28. Theorem: Let (S, +, .) be a simple semiring in which (S, .) is rectangular band then (S, .) is singular.

Proof: Let (S, +.) be a simple semiring and (S.) be a rectangualr band i.e, for any $a, b \in S$ aba = a. Since S is simple, 1+ a = a + 1 = 1, for all $a \in S$. To prove that (S, .) is singular, consider (1 + a)b = 1.b $\Rightarrow$ b + ab + b = a $\Rightarrow$ (b + ab)a = ba $\Rightarrow$ ba + aba = ba $\Rightarrow$ ba + a = ba $\Rightarrow$ (b + 1)a = ba $\Rightarrow$ 1.a = ba $\Rightarrow$ a = ba $\Rightarrow$ ba = a $\Rightarrow$ (S, .) is a right singular.

Again b(1+ a) = b.1 $\Rightarrow$ b + ba = b $\Rightarrow$ a(b + ba) = ab $\Rightarrow$ ab + aba = ab $\Rightarrow$ ab + a = ab $\Rightarrow$ a(b + 1) = ab $\Rightarrow$ a.1 = ab $\Rightarrow$ a = ab $\Rightarrow$ ab = a $\Rightarrow$ (S, .) is left a singular. Therefore (S.) is singular.

1.29. Theorem: Let (S, +.) be a simple semiring in which (S, .) is rectangular band then (S, +) is one of the following:

(i) I-medial (ii) I-semimedial (iii) I-distributive (iv) L-commutative

(v) R-commutative (vi) I-commutative (vii) external commutaitve

(viii) Conditional commutative. (ix) digonal

Proof: Let (S, +.) be a semiring in which (S.) is a rectangular band. Assume that S satisfies the identity $1+a = 1$ for any $a \in S$. Now for any a, b, c, d $\in S$.

(i) Consider $a + b + c + d = a + (b + c) + b$ (by Theorem.1.21. (ii))

$= a + c + b + d$

(S, +) is I- medial.

(ii) Consider $a + a + b + c = a +(a + b) + c = a + (b + a) + c = a + b + a + c \Rightarrow a + a + b + c = a + b + a + c \Rightarrow$ (S, +) is I- left semi medial.

Again $b + c + a + a = b + (c + a) + a = b + (a + c) + a = b + a + c + a \Rightarrow b + c + a + a = b + a + c + a \Rightarrow$ (S, +) is I-right semi medial.

Therefore, (S, +) is I-semi-medial.

(iii) consider $a + b + c = (a) + b + c = a + a + b + c = a + (a + b) + c = a + (b + a) + c = a + b + a + c \Rightarrow$ (S, +) is I-left distributive.

Consider $b + c + a = b + c + (a) = b + c + a + a = b + (c + a) + a =$ $b + (a + c) + a = b + a + c + a \Rightarrow b + c + a = b + a + c + a \Rightarrow$ (S, +) is Iright-distributive. Hence (S, +) is I-distributive.

Similarly we can prove the remaining.

1.30. Theorem: Let (S, +.) be a simple semiring and (S, .) is rectangular band then (S, +) is (i) quasi-seprative (ii) weakly-seperative (iii) seperative.

Proof: Let (S,+, .) be a simple semiring and (S, .) is a rectangular band i.e, for any $a,b \in S$, $aba = a$. Since S is simple, $1+a = a+1 = 1$, for all $a \in S$. Let $a + a = a + b \Rightarrow a + a +1 = a + b + 1 \Rightarrow a + 1 = b + 1 \Rightarrow a = b$. Again, $a + b = b + b \Rightarrow a + b + 1 = b + b + 1 \Rightarrow a + 1 = b + 1 \Rightarrow a = b$. Hence $a + a = a + b = b + b \Rightarrow a = b \Rightarrow$ (S, +) is quasi-seperative.

(ii) Let $a + b = (a) + b = ba + b = b + ab = b + a \Rightarrow a + b = b + a \rightarrow (1)$

From (i) and (ii) $a + a = a + b = b + a = b + b \Rightarrow a = b \Rightarrow$ (S, +) is weakly seperative

(iii) Let $a + a = a + b$

$b + b = b + a$

From (1) $a + b = b +a$ and from theorem 1.20 (S, +) is a band

Therefore, $a = a + a = a + b = b + b = b \Rightarrow a = b$.

Hence (S, +) is seperative.

1.31. Theorem: Let (S, +, .) be a simple semiring in which (S, .) is rectangular band then (S, +) is cancellative in which case $|S| = 1$.

Proof: Let (S, +, .) be a simple semiring in which (S, .) is rectangular band. Since S is simple then for any $a \in S$, $1 + a = a + 1 = 1$.

Let a,b,c, $\in S$. To prove that (S, +) is cancellative, for any a, b, c $\in S$, consider $a + c = b + c$. Then $a + c.1 = b + c.1 \Rightarrow a + c(a + 1)$

$= b + c(b + 1) \Rightarrow a + ca + c = b + cb + c \Rightarrow a + ca + cac = b + cb + cbc$ (since (S, .) is rectangular) $\Rightarrow a + ca(1 + c) = b + cb(1 + c)$

$\Rightarrow a + ca.1 = b + cb.1 \Rightarrow a + ca = b + cb.1 \Rightarrow a + ca = b + cb$

$\Rightarrow (1 + c)a = (1 + c)b \Rightarrow 1.a = 1.b \Rightarrow a = b \Rightarrow a + c = b + c$

$\Rightarrow a = b. \Rightarrow$ (S. +) is right cancellative

Again $c + a = c + b \Rightarrow c.1 + a = c.1 + b \Rightarrow c(1 + a) + a = c(1 + b) + b \Rightarrow c + ca + a = c + cb + b \Rightarrow cac + c + a = cbc + cb + b \Rightarrow cac + ca + a = abc +cb + b \Rightarrow ca(c + 1)+ a = cb(c + 1) + b \Rightarrow ca.1 + a = cb.1 + b$

$\Rightarrow a + a = cb + b \Rightarrow (c + 1)a = (c + 1)b \Rightarrow 1.a = 1.b \Rightarrow a = b \Rightarrow c + a = c + b \Rightarrow a = b \Rightarrow$ (S, +) is left cancellative.

Therefore, (S, +) is cancellative semigroup. Since S is simple semiring we have $1 + a = 1 \Rightarrow 1 + a = 1 + 1$. But (S, +) is cancellative $\Rightarrow a = 1$ for all $a \in S$. Therefore $|S| = 1$.

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