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# On the Analysis of the Finite Element Solutions of Boundary Value Problems Using Gale kin Method

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***ABSTRACT:*** *The finite difference method can be considered as a direct discretization of differential equations but in finite element methods, we generate difference equations by using approximate methods with piecewise polynomial solution. In this paper, we use the Galerkin method to obtain the approximate solution of a boundary value problem. The convergence analysis of these solution are also considered.*

## I. Introduction

Several researchers including Nasser .M. Abbasi [1], S.N. Atluri[2] have shown in general how the finite element method could be used to solve both ordinary and partial differential equations. In this paper, we focused particularly on the Galerkin's method. The Galerkin method is one of the integral parts of the weighted Residual methods. The weighted residual methods are approximate methods which provide analytical procedure for obtaining solution in the form of functions which are close in some sense to the exact solution of the boundary value problem or initial value problem.

Suppose they seek an approximate solution $u(\underset{\sim}{x})$ to a differential equation

$$L\,u(\underset{\sim}{x}) = f(\underset{\sim}{x}) \qquad \underset{\sim}{x} \in \Omega \ (1.1)$$

Where $L$ is a differential operator, $u(\underset{\sim}{x})$ is the unknown solution of interest, $f(\underset{\sim}{x})$is a known forcing function, $\Omega$ is the domain over which the differential equation applies, $\partial\,\Omega$ is the boundary of the domain, $\underset{\sim}{x}$ $is$ a vector containing the independent variables and the proper number and type of boundary conditions are specified along $\partial\,\Omega$.

The aforementioned procedure involves the following steps

i. Choose a trial space $S_N$ of possible approximate solutions. $S_N$ should be a finite N- dimensional function space with a set of basis functions $\{\emptyset_1(\underset{\sim}{x}), \emptyset_2(\underset{\sim}{x}), \dots, \emptyset_N(\underset{\sim}{x})\}$

ii. Pick a trail function $V(\underset{\sim}{x}), \in S_N$, so that

$$V(\underset{\sim}{x}) = \sum_{k=1}^{n} \propto_k \emptyset_k(\underset{\sim}{x}) \qquad (1.2)$$

Where $\propto k$ are scalars

iii. Minimize the residual

$$R(x) = L\,U(x) - f(x) \qquad (1.3)$$

Which is a measure of the amount by which $V(x)$ fails to satisfy the original equation. By doing so, a set of algebraic equations in terms of $\propto_i$ are obtained which are solved to determine the $\propto_s$. The method of weighted residuals (MWR) seeks to minimize $R(x)$ by forcing it to zero in a weighted average sense over the entire domain. This requires choosing a set of linearly independent weight functions $\{w_1(\underset{\sim}{x}), w_2\ (\underset{\sim}{x}),\} \dots, w_m(\underset{\sim}{x})$ such that

$$\int_{\Omega} R(x) w_i\,(\underset{\sim}{x}) d\,\underset{\sim}{x} = o \qquad (1.4)$$

T

herefore, to apply MWR, two choices must be made,

(a) Choice of trial space $S_N$ with concomitant definition of basis functions $\{\phi_{i,}\phi_{2\dots,}\phi_N\}$

(b) Choice of weight functions $\{w_i(\underset{\sim}{x})\}_{i=1}^{m}$

(c) Different choices of weight functions lead to different MWR approximations. Once a weight function $w_i(\underset{\sim}{x})$ is specified, integrations in equation (1.4) can be performed, in conjunction with equation (1.2) and (1.3) to produce a system of algebraic equations in $\propto_i$.

Since our focus is the Galerkin's method, the weight functions are the basis functions for the trial space

$$w_i(x) = \phi_{i\,(x)}$$

$$\therefore \int_{\Omega} R(x)\,\phi_{i\ (x)dx=o}$$

This system can he solved by standard methods. The solution to these equations is substituted into (1.2) to give the approximate solution to the problem. The successive approximations are obtained by increasing N and solving the system again. The convergence of successive approximations gives a clue, but not necessarily a definite one, to the reasonableness of the approximation.

## II. Methodology

**2.1 Example**

Using the Galerkin method, N=3 obtain a finite element solution of the boundary value problem

$$\frac{d^2u}{dx^2} + u = 0 \qquad 0 < x < 1$$

$$u(0) = 1, \qquad u(1) = 0$$

Solution

Let

$$\tilde{u}(x) = \tilde{U_0} + \sum_{i=1}^{N} q_i\,\emptyset_i \qquad (1)$$

be the trial solution

From the boundary condition, $\tilde{U_o} = 1$ and choosing a polynomial $\phi^i(x) = x^i$, equation (1) becomes

$$\hat{U}(x) = 1 + \sum_{i=1}^{3} q_i\,x^i$$

N=3

The residual $R(x)$ becomes

$$R(x) = \frac{d^2\tilde{U}}{d_{x^2}} + \tilde{U} \qquad (2)$$

$$R(x)\frac{d^2}{dx^2}\left[1 + \sum_{i=1}^{3} q_i\,x^i\right] + \left[1 + \sum_{i=1}^{3} qi\ x^i\right]$$

$$R(x) = \frac{d}{dx}\left[\sum_{i=1}^{3} q_i\ (i)x^{i-1}\right] + 1 + \sum_{i=1}^{3} qi\ x^i$$

$$R(x) = \sum_{i=1}^{3} q_i\ ((i)(i-1)x^{i-2} + 1 + \sum_{i=1}^{3} qi\ x^i$$

$$R(x) = 1 + \sum_{i=1}^{3} q_i\ ((i)(i-1)x^{i-2} + x^i)$$

$$R(x) = 1 + q_1x + q_2(2 + x^2) + q_3(6x + x^3)$$

Reducing $R(x)$ to minimum, the integral

$$\int_0^1 R(x)V_i\,(x)dx = 0 \qquad i = 1,2,3$$

We choose the weighted function $V_i(x)$from the family of polynomials that is

$$V_i(x) = x^{i-1}$$

$$\Rightarrow \int_0^1 R(x)\,x^{i-1}dx = 0$$

Thus

$$\int_0^1 \{1 + q_1x + q_2(2 + x^2) + q_3(6x + x^3)\}\,x^{i-1}dx = 0$$

For $i = 1$

$$\int_0^1 \{1 + q_1x + q_2(2 + x^2) + q_3(6x + x^3)\}\,dx = 0$$

$$\left[x + q_1\frac{x^2}{2} + q_2\left(2x + \frac{x^3}{3}\right) + q_3\left(\frac{6x^2}{2} + \frac{x^4}{4}\right)\right]_0^1 = 0$$

$$1 + {}^{1}\!/_{2}\,q_1 + {}^{7}\!/_{3}\,q_2 + {}^{13}\!/_{4}\,q_3 = 0 \qquad (a)$$

For $i = 2$

$$\int_0^1 [1 + q_1 x + q_2(2 + x^2) + q_3(6x + x^3)]x\,dx = 0$$

$$\int_0^1 [x + q_1 x^2 + q_2(2x + x^3) + q_3(6x^2 + x^4)]\,dx = 0$$

$$\frac{x^2}{2} + q_1\frac{x^3}{3} + q_2\left(\frac{2x^2}{2} + \frac{x^4}{4}\right) + q_3\left[\frac{6x^3}{3} + \frac{x^5}{5}\right]_0^1 = 0$$

$$1/_2 + 1/_3\,q_1 + 5/_4\,q_2 + 11/_5\,q_3 = 0 \qquad (b)$$

For $i = 3$

$$\int_0^1 [1 + q_1 x + q_2(2 + x^2) + q_3(6x + x^3)]\,x^2 dx = 0$$

$$\int_0^1 [x^2 + q_1 x^3 + q_2(2x^2 + x^4) + q_3(6x^3 + x^5)]\,dx = 0$$

$$\left[\frac{x^3}{3} + q_1\frac{x^4}{4} + q_2\left(\frac{2x^3}{3} + \frac{x^5}{5}\right) + q_3\left(\frac{6x^4}{4} + \frac{x^6}{6}\right)\right]_0^1 —$$

$$\frac{1}{3} + 1/_4\,q_1 + \frac{13}{15}q_2 + 5/_3\,q_3 = 0 \qquad (c)$$

Solving (a), (b), (c), using crammer's rule (3) yields

$$q_1 = \frac{12}{59},\quad q_2 = -30/_{59},\quad q_3 = \frac{20}{59}$$

Substituting these values into (1) we have

$$\tilde{U}(x) = 1 + \sum_{i=1}^{3} q_i x^i$$

$$= 1 + q_1 x + q_2 x^2 + q_3 x^3$$

$$= 1 + \frac{12}{59}x - \frac{30}{59}x^2 + \frac{20}{59}x^3$$

Note

If N is increased to say 4, successive approximations will be obtained which tends o converge closely to the exact solution

For the exact solution

$$\frac{d^2u}{dx^2} + u = 0 \quad ,0 < x < 1$$

$$u(0) = 1 \quad , \qquad u(1) = 0$$

Solution

Auxiliary equation

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$\therefore \quad u(x) = A\,Cos\,x + Bsin\,x$$

Using the boundary condition

$$u(0) = 1 = ACos\,0 + Bsin\,0$$

$$A = 1$$

$$u(1) = 0 = Acos(1) + Bsin\,(1)$$

$$0 = 1(0.540) + B(0.841)$$

$$B = -0.642$$

$$\therefore \qquad u(x) = Cos\,x - 0.642\,sin\,x$$

Comparison of results

Considering the nodal point $\quad x = 1/_2 = 0.5$

$$u(0.5) = \cos(0.5) - 0.642\sin(0.5)$$

$$= 0.878 - 0.642\,(0.479)$$

$$= 0.878 - 0.308$$

$$= 0.570$$

But for the approximate solution

$$\tilde{U}(0.5) = 1 + \frac{12}{59}(0.5) - \frac{30}{59}(0.5)^2 + \frac{20}{59}(0.5)^3$$
$$= 1.017$$

Therefore $error$ $= 0.570 - 1.017$
$$= -0.447$$

2.2 Example

Given the boundary value problems

$$\frac{dy}{dx} - y(x) = 0$$

Defined over $0 \leq x \leq 1$ with the boundary condition $y(0) = 1$, $N = ,3,$ solve the $ODE$ using the Galerkm method.

Solution

We assume a solution that is valid over the domain $0 \leq x \leq 1$ of the form

$$\tilde{y}(x) = \tilde{y_o} + \sum_{j=1}^{3} q_j \emptyset_j (x)$$

This trial solution has $\acute{y}_0 = 1$ at the initial condition $x = 0$, and choosing $\emptyset_j(x) = x^j$ from the family of polynomials hence our trial solution becomes.

$$\tilde{y} = 1 + \sum_{j=1}^{3} q_j x^j$$

Now we need to determine the coefficients $q_j$, and then our solution will be complete

Substituting this solution $\tilde{y}(x)$into the original ODE (1), we obtain the residue

$$R(x) = \frac{d\tilde{y}}{dx} - \tilde{y}(x)$$

This is the error which will result when the assumed solution is used in place of the exact solution

Hence from (1), we find the residual to be

$$R(x) = \frac{d}{dx}\left(1 + \sum_{j=1}^{3} q_j x^j\right) - \left(1 + \sum_{j=1}^{3} q_j x^j\right)$$
$$= \sum_{j=1}^{3} q_j jx^{j-1} - \left(1 + \sum_{j=1}^{3} q_j x^j\right)$$
$$= -1 + \sum_{j=1}^{3} q_j (jx^{j-1} - x^j) \qquad (2)$$
$$= -1 + q_1(1-x) + q_2(2x - x^2) + q_3(3x^2 - x^3)$$

Our goal now is to reduce this residual to minimum. The way we achieve this is by requiring that the residual satisfies the following integral equation

$$\int_\Omega v_i R(x) dx = 0 \qquad (3)$$
$$i = 1, \ldots, N$$

The above is a set of N equations, The integration is carried over the whole domain, and $v_i(x)$ is a weight (test) function selected from the family of polynomial since we are using the Galerkm method

That is

$$v_i(x) = x^{i-1}$$

Substitute the above in (3) we obtain

$$\int_{x=0}^{x=1} x^{i-1} R(x) dx = 0 \qquad i - 1,2,3$$

$$\int_0^1 x^{i-1}(-1 + q_1(1 - x^i) + q_2(2x - x^2) + q_3(3x^3 - x^3))dx = 0$$

$For\ i = 1$

$$\int_0^1 -1 + q_1(1 - x^1) + q_2(2x - x^2) + q_3(3x^2 - x^3))dx = 0$$

$For\ i = 2$

$$\int_0^1 x(-1 + q_1(1 - x^1) + q_2(2x - x^2) + q_3(3x^2 - x^3))dx = 0$$

$For\ i = 3$

$$\int_0^1 x^2(-1 + q_1(1 - x^1) + q_2(2x - x^2) + q_3(3x^2 - x^3))dx = 0$$

Now carrying the integration above, we obtain the following three equations.

$$- \quad 1 + \tfrac{1}{2}\, q_1 \ + \tfrac{2}{3}\, q_2 \ + \tfrac{3}{4}\, q_3 + 0 \quad (a)$$

$$- \quad \tfrac{1}{2} + \tfrac{1}{6}\, q_1 \ + \tfrac{5}{2}\, q_2 \ + \tfrac{11}{20}\, q_3 + 0 \qquad (b)$$

$$-\frac{1}{3} + \frac{1}{12} q_1 \ + \ \frac{3}{10}\, q_2 \ + \frac{13}{30}\, q_3 + 0 \qquad (c)$$

Solving (a), (b), (c) using crammer's rule yields

$$q_1 = \frac{72}{71}, q_2 = \frac{30}{71}, q_3 = \frac{20}{71}$$

Hence our assumed series solution is now complete, using the above coefficient and from equation (1) we write.

$$\tilde{y} = 1 + \sum_{J=1}^{3} q_j\, x^j$$

$$\tilde{y} = 1 + \ q_1 x + q_2 x^2 + q_3 x^3$$

Hence

$$\tilde{y}(x) = 1 + \frac{22}{71} x + \frac{30}{71} x^2 + \frac{20}{71} x^3$$

The exact solution is $y = e^x$

**Convergence analysis**

The accuracy in the finite element solution can be increased either by decreasing the size of the elements or by increasing the degree of the polynomial in the piece wise approximate solution. The convergence of the finite element solution to the exact solution as the size of the finite element approaches zero is obtained if the following conditions are satisfied. [4]

**Completeness**

This is the condition that, as the size of the finite element approaches zero, the terms occurring under the integral sign in the weighted residual must tend to be single valued and well behaved, thus the set of shape functions chosen must be able to represent any constant value of the function u as well as the derivatives up to order m within each element in the limit as element size approaches zero.

**Compatibility**

This is an inter element continuity condition. If the order of the highest derivative in the weighted residual equation $R(x)w_i(x)$ is m, then the finite elements and the shape functions are to be selected such that at the element interfaces, the function $U$ has continuity of all the derivatives up to the order $m - 1$.

The finite elements satisfying this criterion are called the conforming dements, otherwise non conforming elements.

## III. Conclusion

We observed that the error $E\ (x) = u(x) - \tilde{V}\ (x)$ gotten at certain nodal points could be minimized if we increase the value of N to obtain successive approximations of the original solution

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