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# Fractional Integration Operators of Certain Generalalized Hyper geometric Function of Two Variables

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***Abstract:*** In the present paper the authors introduce two new fractional integration operators associated with $\overline{H}$-function of two variables. Three important properties of these operators are established which are the generalization of the results given earlier by several authors.

***Key words:*** Fractional Integration Operators, $\overline{H}$-function of two variables, Mellin Transform (2000 Mathematics subject classification : 33C99)

## I. Introduction

The $\overline{H}$-function of two variables defined and represented by Singh and Mandia [10] in the following manner:

$$\overline{H}[x,y]=\overline{H}\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]=\overline{H}^{o,n_1:\ m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right]$$

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\phi_2(\xi)\phi_3(\eta)x^{\xi}y^{\eta}d\xi d\eta \qquad (1.1)$$

Where

$$\phi_1(\xi,\eta)=\frac{\prod_{j=1}^{n_1}\Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)}{\prod_{j=n_1+1}^{p_1}\Gamma\left(a_j-\alpha_j\xi-A_j\eta\right)\prod_{j=1}^{q_1}\Gamma\left(1-b_j+\beta_j\xi+B_j\eta\right)} \qquad (1.2)$$

$$\phi_2(\xi)=\frac{\prod_{j=1}^{n_2}\left\{\Gamma\left(1-c_j+\gamma_j\xi\right)\right\}^{K_j}\prod_{j=1}^{m_2}\Gamma\left(d_j-\delta_j\xi\right)}{\prod_{j=n_2+1}^{p_2}\Gamma\left(c_j-\gamma_j\xi\right)\prod_{j=m_2+1}^{q_2}\left\{\Gamma\left(1-d_j+\delta_j\xi\right)\right\}^{L_j}} \qquad (1.3)$$

$$\phi_3(\eta)=\frac{\prod_{j=1}^{n_3}\left\{\Gamma\left(1-e_j+E_j\eta\right)\right\}^{R_j}\prod_{j=1}^{m_3}\Gamma\left(f_j-F_j\eta\right)}{\prod_{j=n_3+1}^{p_3}\Gamma\left(e_j-E_j\eta\right)\prod_{j=m_3+1}^{q_3}\left\{\Gamma\left(1-f_j+F_j\eta\right)\right\}^{S_j}} \qquad (1.4)$$

Where $x$ and $y$ are not equal to zero (real or complex), and an empty product is interpreted as unity $p_i,q_i,n_i,m_j$ are non-negative integers such that $0\le n_i\le p_i, o\le m_j\le q_j (i=1,2,3; j=2,3)$. All the $a_j(j=1,2,...,p_1), b_j(j=1,2,...,q_1), c_j(j=1,2,...,p_2), d_j(j=1,2,...,q_2)$, $e_j(j=1,2,...,p_3), f_j(j=1,2,...,q_3)$ are complex parameters.

$\gamma_j\ge 0(j=1,2,...,p_2), \delta_j\ge 0(j=1,2,...,q_2)$ (not all zero simultaneously), similarly $E_j\ge 0(j=1,2,...,p_3), F_j\ge 0(j=1,2,...,q_3)$ (not all zero simultaneously). The exponents $K_j(j=1,2,...,n_3), L_j(j=m_2+1,...,q_2), R_j(j=1,2,...,n_3), S_j(j=m_3+1,...,q_3)$ can take on non-negative values.

The contour $L_1$ is in $\xi$-plane and runs from $-i\infty$ to $+i\infty$. The poles of $\Gamma\left(d_j-\delta_j\xi\right)(j=1,2,...,m_2)$ lie to the right and the poles of $\Gamma\left\{\left(1-c_j+\gamma_j\xi\right)\right\}^{K_j}$ $(j=1,2,...,n_2),\Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)(j=1,2,...,n_1)$ to the left of the contour. For $K_j(j=1,2,...,n_2)$ not an integer, the poles of gamma functions of the numerator in (1.3) are converted to the branch points.

The contour $L_2$ is in $\eta$-plane and runs from $-i\infty$ to $+i\infty$. The poles of $\Gamma\left(f_j-F_j\eta\right)(j=1,2,...,m_3)$ lie to the right and the poles of $\Gamma\left\{\left(1-e_j+E_j\eta\right)\right\}^{R_j}$ $(j=1,2,...,n_3),\Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)(j=1,2,...,n_1)$ to the left of the contour. For $R_j(j=1,2,...,n_3)$ not an integer, the poles of gamma functions of the numerator in (1.4) are converted to the branch points.

The functions defined in (1.1) is an analytic function of $x$ and $y$, if

$$U=\sum_{j=1}^{p_1}\alpha_j+\sum_{j=1}^{p_2}\gamma_j-\sum_{j=1}^{q_1}\beta_j-\sum_{j=1}^{q_2}\delta_j<0 \quad (1.5)$$

$$V=\sum_{j=1}^{p_1}A_j+\sum_{j=1}^{p_3}E_j-\sum_{j=1}^{q_1}B_j-\sum_{j=1}^{q_3}F_j<0 \quad (1.6)$$

The integral in (1.1) converges under the following set of conditions:

$$\Omega=\sum_{j=1}^{n_1}\alpha_j-\sum_{j=n_1+1}^{p_1}\alpha_j+\sum_{j=1}^{m_2}\delta_j-\sum_{j=m_2+1}^{q_2}\delta_jL_j+\sum_{j=1}^{n_2}\gamma_jK_j-\sum_{j=n_2+1}^{p_2}\gamma_j-\sum_{j=1}^{q_1}\beta_j>0 \quad (1.7)$$

$$\Lambda=\sum_{j=1}^{n_1}A_j-\sum_{j=n_1+1}^{p_1}A_j+\sum_{j=1}^{m_2}F_j-\sum_{j=m_2+1}^{q_2}F_jS_j+\sum_{j=1}^{n_3}E_jR_j-\sum_{j=n_2+1}^{p_3}E_j-\sum_{j=1}^{q_1}B_j>0 \quad (1.8)$$

$$|\arg x|<\frac{1}{2}\Omega\pi,\ |\arg y|<\frac{1}{2}\Lambda\pi \quad (1.9)$$

The behavior of the $\overline{H}$-function of two variables for small values of $|z|$ follows as:

$$\overline{H}[x,y]=0(|x|^{\alpha}|y|^{\beta}),\max\{|x|,|y|\}\to 0 \quad (1.10)$$

Where

$$\alpha=\min_{1\le j\le m_2}\left[\operatorname{Re}\left(\frac{d_j}{\delta_j}\right)\right] \qquad \beta=\min_{1\le j\le m_2}\left[\operatorname{Re}\left(\frac{f_j}{F_j}\right)\right] \quad (1.11)$$

For large value of $|z|$,

$$\overline{H}[x,y]=0\left\{|x|^{\alpha'},|y|^{\beta'}\right\},\min\{|x|,|y|\}\to 0 \quad (1.12)$$

Where

$$\alpha'=\max_{1\le j\le n_2}\operatorname{Re}\left(K_j\frac{c_j-1}{\gamma_j}\right),\ \beta'=\max_{1\le j\le n_3}\operatorname{Re}\left(R_j\frac{e_j-1}{E_j}\right) \quad (1.13)$$

Provided that $U<0$ and $V<0$.

If we take

$K_j=1(j=1,2,...,n_2),L_j=1(j=m_2+1,...,q_2),R_j=1(j=1,2,...,n_3),S_j=1(j=m_3+1,...,q_3)$ in (2.1), the $\overline{H}$-function of two variables reduces to $H$-function of two variables due to [7].

If we set $n_1=p_1=q_1=0$, the $\overline{H}$-function of two variables breaks up into a product of two $\overline{H}$-function of one variable namely

$$\overline{H}^{0,0:m_2,n_2:m_3,n_3}_{0,0:p_2,q_2:p_3,q_3}\left[\begin{array}{c|l} x & —:\left(c_j,\gamma_j;K_j\right)_{1,n_2},\left(c_j,\gamma_j\right)_{n+1,p_2}:\left(e_j,E_j;R_j\right)_{1,n_3},\left(e_j,E_j\right)_{n_3+1,p_3} \\ y & —:\left(d_j,\delta_j\right)_{1,m_2},\left(d_j,\delta_j;L_j\right)_{m_2+1,q_2}:\left(f_j,F_j\right)_{1,m_3},\left(f_j,F_j;S_j\right)_{m_3+1,q_3}\end{array}\right]$$

$$= \overline{H}^{m_2,n_2}_{p_2,q_2}\left[x\left|\begin{matrix}(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}\\(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2}\end{matrix}\right.\right]\overline{H}^{m_3,n_3}_{p_3,q_3}\left[y\left|\begin{matrix}(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right] \quad (1.14)$$

If $\lambda > 0$, we then obtain

$$\lambda^2\overline{H}^{0,n_1:\, m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x^\lambda\\y^\lambda\end{matrix}\left|\begin{matrix}(a_j,\lambda\alpha_j;A_j)_{1,p_1},(c_j,\lambda\gamma_j;K_j)_{1,n_2},(c_j,\lambda\gamma_j)_{n_2+1,p_2},(e_j,\lambda E_j;R_j)_{1,n_3},(e_j,\lambda E_j)_{n_3+1,p_3}\\(b_j,\lambda\beta_j;B_j)_{1,q_1},(d_j,\lambda\delta_j)_{1,m_2},(d_j,\lambda\delta_j;L_j)_{m_2+1,q_2},(f_j,\lambda F_j)_{1,m_3},(f_j,\lambda F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]$$

$$= \overline{H}^{0,n_1:\, m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right] \quad (1.15)$$

$$\overline{H}^{0,n_1:\, m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}1/x\\1/y\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]$$

$$= \overline{H}^{0,n_1:\, m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}(1-b_j,\beta_j;B_j)_{1,q_1},(1-d_j,\delta_j)_{1,m_2},(1-d_j,\delta_j;L_j)_{m_2+1,q_2},(1-f_j,F_j)_{1,m_3},(1-f_j,F_j;S_j)_{m_3+1,q_3}\\(1-a_j,\alpha_j;A_j)_{1,p_1},(1-c_j,\gamma_j;K_j)_{1,n_2},(1-c_j,\gamma_j)_{n_2+1,p_2},(1-e_j,E_j;R_j)_{1,n_3},(1-e_j,E_j)_{n_3+1,p_3}\end{matrix}\right.\right] \quad (1.16)$$

## II. Definitions

We introduce the fractional integration operators by means of the following integral equations:

$$Y\left[\begin{smallmatrix}\delta,\beta;r\\\lambda,\mu;x\end{smallmatrix} f(x)\right] = rx^{-\delta-r\beta-1}\int_0^x t^\delta\left(x^r-t^r\right)^\beta f(t)\overline{H}\left[\begin{smallmatrix}\lambda U\\\mu U\end{smallmatrix}\right]dt \quad (2.1)$$

And

$$N\left[\begin{smallmatrix}\alpha,\beta;r\\\lambda,\mu;x\end{smallmatrix} f(x)\right] = rx^{\alpha}\int_0^x t^{-\alpha-r\beta-1}\left(t^r-x^r\right)^\beta f(t)\overline{H}\left[\begin{smallmatrix}\lambda V\\\mu V\end{smallmatrix}\right]dt \quad (2.2)$$

Where $U$ and $V$ represent the expressions

$\left(\frac{t^r}{x^r}\right)^m\left(1-\frac{t^r}{x^r}\right)^n$ and $\left(\frac{x^r}{t^r}\right)^m\left(1-\frac{x^r}{t^r}\right)^n$

Respectively, $r, m, n$ are positive integers and

$|\arg\lambda| < \frac{1}{2}\Omega\pi,\ |\arg\mu| < \frac{1}{2}\Lambda\pi$.

The conditions of the validity of these operators are as follow:

(i) $1 \le p, q < \infty, \frac{1}{p}+\frac{1}{q}=1$

(ii) $\mathrm{Re}\left(\delta + rm\frac{d_j}{\delta_j} + rm\frac{f_j}{F_j}\right) > -\frac{1}{q}$,

$\mathrm{Re}\left(\beta + rn\frac{d_j}{\delta_j} + rn\frac{f_j}{F_j}\right) > -\frac{1}{q}$,

$\mathrm{Re}\left(\alpha + rm\frac{d_j}{\delta_j} + rm\frac{f_j}{F_j}\right) > -\frac{1}{p}$,

(iii) $f(x) \in L_p(0,\infty)$.

The last condition ensures that $Y$ and $N$ both exists and also that both belongs to $L_p(0,\infty)$.

If we set $n_1 = p_1 = q_1 = 0$, we obtain the operators involving the product of two $\overline{H}$ -functions.

If we take

$K_j = 1 (j = 1,2,...,n_2), L_j = 1 (j = m_2+1,...,q_2), R_j = 1 (j = 1,2,...,n_3), S_j = 1 (j = m_3+1,...,q_3)$ we obtain the operators involving $H$ -function of two variables.

## III. Mellin Transform

The Mellin transform of $f(t)$ will be defined by $M[f(t)]$. We write $s = p^{-1} + it$ where $p$ and $t$ are real. If $p \geq 1, f(t) \in L_p(0,\infty)$, then

$$p = 1,\ M[f(t)] = \int_0^\infty t^{s-1} f(t)dt \qquad (3.1)$$

$$p > 1,\ M[f(t)] = l.i.m. \int_{1/x}^{x} t^{s-1} f(t)dt \qquad (3.2)$$

Where $l.i.m.$ denotes the usual limit in the mean for $L_p$-spaces.

**Theorem 1.** If

$$f(x) \in L_p(0,\infty), 1 \leq p \leq 2[or\ f(x) \in M_p(0,\infty) and\ p > 2] |\arg\lambda| < \frac{1}{2}\Omega\pi, |\arg\mu| < \frac{1}{2}\Lambda\pi, (\Omega,\Lambda) > 0$$

$$\mathrm{Re}\left(\delta + rm\frac{d_j}{\delta_j} + rm\frac{f_j}{F_j}\right) > -\frac{1}{q},\ \mathrm{Re}\left(\beta + rn\frac{d_j}{\delta_j} + rn\frac{f_j}{F_j}\right) > -\frac{1}{q}, \text{ then}$$

$$M\left\{Y\left[{}^{\delta,\beta;r}_{\lambda,\mu;x} f(x)\right]\right\} = \overline{H}^{o,n_1:\quad m_2,n_2:m_3,n_2}_{p_1+2,q_1+2:p_2,q_2;p_2,q_2}$$

$$\left[{}^{x}_{y}\left|{}^{(a_j,\alpha_j;A_j)_{1,p_1},\left(\frac{-\delta+s}{r},m\right),(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1},\left(-\beta-\frac{1+\delta-s}{r},m+n\right),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}}\right.\right] M[f(x)] \qquad (3.3)$$

Where $M_p(0,\infty)$ denotes the class of all functions $f(x)$ of $L_p(0,\infty)$ with $p > 2$ which are inverse Mellin transforms of functions of $L_q(-\infty,\infty)$.

**Proof:** From (2.1), it follows that

$$M\left\{Y\left[{}^{\delta,\beta;r}_{\lambda,\mu;x} f(x)\right]\right\} = \int_0^\infty x^{s-1} r x^{-\delta-r\beta-1} \int_0^x t^\delta \left(x^r - t^r\right)^\beta f(t)\overline{H}\left[{}^{\lambda U}_{\mu U}\right] dt dx$$

$$= \int_0^\infty t^\delta f(t)dt \int_0^x x^{s-\delta-r\beta-2}\left(x^r - t^r\right)^\beta f(t)\overline{H}\left[{}^{\lambda U}_{\mu U}\right] dx,$$

The change of order of integration is permissible by virtue of de la Vallee Poisson's theorem ([2],p.504)under the conditions stated with the theorems. The theorem readily follows on evaluating the $x$-integral by means of the formula

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\overline{H}^{o,n_1:\ m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[{}^{\lambda x^k(1-x)^l}_{\mu x^k(1-x)^l}\left|{}^{(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}}\right.\right]dx$$

$$= \overline{H}^{o,n_1:\quad m_2,n_2:m_3,n_2}_{p_1+2,q_1+2:p_2,q_2;p_2,q_2}\left[{}^{\lambda}_{\mu}\left|{}^{(a_j,\alpha_j;A_j)_{1,p_1},(1-\alpha,k),(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}}_{(b_j,\beta_j;B_j)_{1,q_1},(1-\alpha-\beta,k+l),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}}\right.\right] \qquad (3.4)$$

Where

$$\mathrm{Re}\left(\alpha + k\frac{d_j}{\delta_j} + k\frac{f_j}{F_j}\right) > 0, \mathrm{Re}\left(\beta + k\frac{d_j}{\delta_j} + k\frac{f_j}{F_j}\right) > 0\ |\arg\lambda| < \frac{1}{2}\Omega\pi, |\arg\mu| < \frac{1}{2}\Lambda\pi, (\Omega,\Lambda) > 0,$$

which follows from the definition of the $\overline{H}$-function of two variables and Beta function formula.

In a similar manner, the following theorems can be establisded.

**Theorem 2.** If

$$f(x) \in L_p(0,\infty), 1 \leq p \leq 2[or\ f(x) \in M_p(0,\infty) and\ p > 2] |\arg\lambda| < \frac{1}{2}\Omega\pi, |\arg\mu| < \frac{1}{2}\Lambda\pi, (\Omega,\Lambda) > 0$$

$$\mathrm{Re}\left(\beta + rn\frac{d_j}{\delta_j} + rn\frac{f_j}{F_j}\right) > -\frac{1}{q},\ \mathrm{Re}\left(\alpha + \beta r + rm\frac{d_j}{\delta_j} + rm\frac{f_j}{F_j}\right) > -\frac{1}{q}, \frac{1}{p} + \frac{1}{q} = 1 \text{ then}$$

$$M\left\{N\left[\begin{smallmatrix}\alpha,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right]\right\} = \overline{H}^{o,n_1:\quad m_2,n_2:m_3,n_2}_{p_1+2,q_1+2:p_2,q_2;p_2,q_2}$$

$$\left[\begin{matrix}\lambda\\ \mu\end{matrix}\left|\begin{matrix}\left(a_j,\alpha_j;A_j\right)_{1,p_1},\left(\frac{1-\alpha\delta-s}{r},m\right),\left(c_j,\gamma_j;K_j\right)_{1,n_2},\left(c_j,\gamma_j\right)_{n_2+1,p_2},\left(e_j,E_j;R_j\right)_{1,n_3},\left(e_j,E_j\right)_{n_3+1,p_3}\\ \left(b_j,\beta_j;B_j\right)_{1,q_1},\left(-\beta-\frac{\alpha+s}{r},m+n\right),\left(d_j,\delta_j\right)_{1,m_2},\left(d_j,\delta_j;L_j\right)_{m_2+1,q_2},\left(f_j,F_j\right)_{1,m_3},\left(f_j,F_j;S_j\right)_{m_3+1,q_3}\end{matrix}\right.\right] M[f(x)] \qquad (3.5)$$

**Theorem 3.** If $f(x)\in L_p(0,\infty), g(x)\in L_p(0,\infty), |\arg\lambda| < \frac{1}{2}\Omega\pi, |\arg\mu| < \frac{1}{2}\Lambda\pi, (\Omega,\Lambda)>0$,

$\mathrm{Re}\left(\beta + rm\frac{d_j}{\delta_j} + rm\frac{f_j}{F_j}\right) > \max\left(\frac{1}{p},\frac{1}{q}\right)$, $\mathrm{Re}\left(\beta r + rm\frac{d_j}{\delta_j} + rm\frac{f_j}{F_j}\right) > 0, \frac{1}{p}+\frac{1}{q}=1$ then

$$\int_0^\infty g(x) Y\left[\begin{smallmatrix}\delta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right]dx = \int_0^\infty f(x) N\left[\begin{smallmatrix}\delta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} g(x)\right]dx \qquad (3.6)$$

## IV. Formal Properties of the Operators

Here, we give some formal properties of the operators which follow as consequences of the definitions (2.1) and (2.2).

$$x^{-1} Y\left[\begin{smallmatrix}\delta,\beta;r\\ \lambda,\mu;x^{-1}\end{smallmatrix} f(x^{-1})\right] = N\left[\begin{smallmatrix}\delta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right] \qquad (4.1)$$

$$x^{-1} N\left[\begin{smallmatrix}\alpha,\beta;r\\ \lambda,\mu;x^{-1}\end{smallmatrix} f(x^{-1})\right] = Y\left[\begin{smallmatrix}\alpha,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right] \qquad (4.2)$$

$$x^{\eta} Y\left[\begin{smallmatrix}\delta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right] = Y\left[\begin{smallmatrix}\delta-\eta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} x^{\eta} f(x)\right] \qquad (4.3)$$

$$x^{\eta} N\left[\begin{smallmatrix}\alpha,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right] = N\left[\begin{smallmatrix}\alpha+\eta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} x^{\eta} f(x)\right] \qquad (4.4)$$

If

$Y\left[\begin{smallmatrix}\delta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right] = g(x)$, then

$$Y\left[\begin{smallmatrix}\delta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(cx)\right] = g(cx) \qquad (4.5)$$

And if

$N\left[\begin{smallmatrix}\alpha,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right] = \phi(x)$, then

$$N\left[\begin{smallmatrix}\alpha,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(cx)\right] = \phi(cx) \qquad (4.6)$$

In conclusion, it is interesting to observe that double integral operators associated with $\overline{H}$ -function of two variables can be defined in the same way and their various properties, analogous to the integral operators studied in this paper can be obtained.

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