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The stochastic exponential population growth model with mixture noise

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ABSTRACT:*ThisWe know that the growth rate at time t is not completely definite and it depends on some random environment effects. So, we consider the stochastic exponential population growth model. We suppose the noise in the population growth model be the mixture noise. The main purpose is to analyze the effect of the mixture noise perturbations as Gaussian process on the growth rate. The expectations and variances of solutions are obtained. However, the confidence interval for the solution of stochastic exponential population growth model where the so-called parameter, population growth rate is not completely definite and it depends on some random environmental effects is obtained. For a case study, we consider the population growth of Iran and obtain the output of models for this data and predict the population individuals in each year.*

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I. INTRODUCTION

Population growth is the change in population over time. Environmental scientists use two models to describe how populations grow over time, the exponential growth model and the logistic growth model. In exponential growth, the population size increases at an exponential rate over time.

As such as, the growth rate at time t is not completely definite and it depends on some random environment effects. In 1969, Levins showed the effect of random variation of different types on population growth[1]. Capocelli and Ricciardi described a diffusion model for population growth in a random environment[2]. Kamel showed the numerical approximations for population growth models[7].

Braumann[3] proposed the applications of stochastic differential equations to population growth. Matisa and Kiffe[4], Andreis and Ricci[5] used of the stochastic exponential population growth model in their studies. The numerical methods for the exponential model of growth proposed by Liu[6].

We know, the growth rate is depended to many different random environment effect. So, in this here, we let that the this random effects were to the linear combination of some white noise. Then, we consider the perturbation effects the mixture noise on the growth rate of population model. However, we will predict the Iran population in 2020.

The organization of this paper is as follows: In this next section, we will define the calculus stochastic and mixture noise. In section 3, we will consider the stochastic exponential population growth model with mixture noise. In section 4, We construct a confidence interval for number of population obtained. The numerical calculus is given in section 5.

II. PRELIMINARIES

There are two main stochastic calculus, Ito and Stratonovich calculus. They yield different solutions and even qualitatively different predictions. In this here, we consider the Ito calculus for random population growth rate model. The goal of this section is to recall notations and definition of the Ito integral and stochastic differential equation that are important for this paper.

Definition 1.

A stochastic process, $\{N_t : 0 \le t \le \infty\}$, is a standard Brownian motion if:

1. $N_0 = 0,$
2. It has continuous sample paths,
3. It has independent, normally distributed increments.

Let $0 = t_0 \prec t_1 \prec t_2 \prec \cdots \prec t_{n-1} \prec t_n = T$ be a grid of points on the interval [0, T]. We know the Riemann integral is defined as a limit

$$\int_0^T f(x)dx = Lim_{\Delta t \to 0} \sum_{i=1}^{n} f(t_i')\,\Delta t_i$$

where $\Delta t_i = t_i - t_{i-1}$ and $t_{i-1} \le t_i' \le t_i$. Similarly, the Ito integral is the limit:

$$\int_0^T f(t)\,dW_t = Lim_{\Delta t \to 0} \sum_{i=1}^{n} f(t_{i-1})\Delta W_i$$

where $\Delta W_i = W(t_i) - W(t_{i-1}),$ a step of Brownian motion across the interval.

The differential dI is a notional convenience, thus, $I = \int_0^T f(t)\,dW_t$ is expressed in differential form as $dI = f\ dW_t$. The differential dW_t of Brownian motion W_t is called white noise.

Definition 2.

A diffusion process is modeled as a differential equation involving deterministic, or drift terms, and stochastic, or diffusion terms, the latter represented by a wiener process, as in the equation:

$$dX_t = f(t, X_t)\,dt + g(t, X_t)\,dW_t, \qquad (1)$$

or the form integral equation is:

$$X(t) = X(0) + \int_0^T f(s, X_s)\,ds + \int_0^T g(s, X_s)\,dW_s. \qquad (2)$$

The equation (1) is the stochastic differential equation (SDE) and the meaning of the last integral in (2) is called the Ito integral.

Definition 3.

A mixture noise may be interpreted as any linear combinationof Wiener processes. The process X_t is a mixture noise if it satisfy the linear additive SDE:

$$dX_t = \sum_{k=1}^{n} \alpha_k\, W_k(t), \qquad \sum_{k=1}^{n} \alpha_k = 1, \qquad (3)$$

where $W_k(t) = \frac{dB_k(t)}{dt}$ are one dimensional white noise processes, $B_k(t)$ are the one dimensional Brownian motion and α_k are constants.

III. STOCHASTIC EXPONENTIAL POPULATION GROWTH MODEL WITH MIXTURE NOISE

Let N=N(t) be the size at time $t \ge 0$ of a population. However, we assume $\frac{dN}{dt}$ be the total growth rate and to the per capita growth rate $a_t = \frac{1}{N}\frac{dN}{dt}$ simply by growth rate.

Consider the following simple population growth model:

$$\begin{cases} \frac{dN(t)}{dt} = a(t)\,N(t) \\ N(0) = N_0 \end{cases} \qquad (4)$$

where N_0 is the initial number at time $t = 0$ and $a(t)$ is the growth rate at time t.

If $a(t) = r(t)$ be the nonrandom function, then

$$N(t) = N_0 \exp\left(\int_0^t r(s)\,ds\right).$$

In special case, if $r(t) = r$, we get $N(t) = N_0 \exp(rt).$

Now, suppose that $a(t)$ depends on some random environment effects, i.e. $a(t) = r(t) +$ " mixture noise", where $r(t)$ is a nonrandom function. Let,

$$\text{" mixture noise"} = \sum_{k=1}^{n} \alpha_k W_k(t)$$

Theorem 1. Let:

$$\begin{cases} \dfrac{dN(t)}{dt} = (r(t) + \sum_{k=1}^{n} \alpha_k \dfrac{dB_k(t)}{dt}) N(t) \\ N(0) = N_0, \end{cases} \tag{5}$$

be stochastic exponential model, then the solution is given by

$$N(t) = N_0 \exp(\int_0^t \left[r(s) - \frac{1}{2} \sum_k \int \alpha_k^2(s) \right] ds + \sum_{k=1}^{n} \int_0^t \alpha_k(s) dB(s)) \tag{6}$$

Proof. By (5), we have

$$\begin{cases} \dfrac{dN(t)}{dt} = (r(t) + \sum_{k=1}^{n} \alpha_k \dfrac{dB_k(t)}{dt}) N(t) \\ \int_0^t \dfrac{dN(s)}{N(s)} = \int_0^t r(s) ds + \sum_{k=1}^{n} \alpha_k \int_0^t dB(s) \end{cases} \tag{7}$$

By using of the 1-dimensional Ito formula [1] and $g(t, x) = \ln x,$ we get,

$$d(\ln N(t)) = \frac{dN(t)}{N(t)} - \frac{1}{2N^2(t)} (r(t) N(t) dt + N(t) \sum \alpha_k dB_k(t))^2$$

$$= \frac{dN(t)}{N(t)} - \frac{1}{2N^2(t)} \left[N^2(t) \sum_k \alpha_k^2 dt \right]$$

because $dt.dB_t = dB_t.dt = dt.dt = 0$ [1], and $dB_i.dB_j = \delta_{ij} dt,$ where, $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$, so

$(dX(t))^2 = \sum_{k=1}^{n} \alpha_k^2 dt$, and

$$d(\ln N(t)) = \frac{dN(t)}{N(t)} - \frac{1}{2N^2(t)} \left[N^2(t) \sum_k \alpha_k^2 dt \right]$$

$$d(\ln N(t)) = \frac{dN(t)}{N(t)} - \frac{1}{2} \sum_k \alpha_k^2(t) dt$$

or,

$$\int_0^t \frac{dN(s)}{N(s)} = \ln \frac{N(t)}{N(0)} + \frac{1}{2} \sum_k \int \alpha_k^2(s) ds. \tag{8}$$

By combination of (7) and (8), we have:

$$\int_0^t r(s)\,ds+\sum_{k=1}^{n}\int_0^t \alpha_k(s)\,dB(s)=\ln\frac{N(t)}{N(0)}+\frac{1}{2}\sum_k \alpha_k^2(s)\,ds$$

$$\Rightarrow \ln\frac{N(t)}{N_0}=\int_0^t\left[\,r(s)-\frac{1}{2}\sum_k\int\alpha_k^2(s)\,\right]ds+\sum_{k=1}^{n}\int_0^t \alpha_k(s)\,dB(s)$$

or,

$$N(t)=N_0.\exp\,(\int_0^t\left[\,r(s)-\frac{1}{2}\sum_k\int\alpha_k^2(s)\,\right]ds+\sum_{k=1}^{n}\int_0^t \alpha_k(s)\,dB(s)). \tag{9}$$

which establish theorem. In $r(t)=r$ and $\alpha(t)=\alpha$ (r and α are constant values), we get:

$$N(t)=N_0.\exp(\left[r-\frac{1}{2}\sum_k\alpha_k^2\right]t+B_t\sum_{k=1}^{n}\alpha_k\,). \tag{10}$$

Theorem 2. In (5), if N_0 and $B_k(t)(k=1,2,\cdots,n)$ be independent random variables, then the expected value and variance of $N(t)$ is:

$$E(\,N(t))=E(\,N_0)\exp(\int_0^t r(s)\,ds),$$

$$Var(\,N(t))=\exp(2\int_0^t r(s)\,ds)\left\{(Var(\,N_0)+E^2(N_0))\exp\,(\sum_{k=1}^{n}\int_0^t\alpha^2(s)\,ds)-E^2(N_0)\right\}. \tag{11}$$

Proof. From (9), we have:

$$N(t)=E(N_0).\exp\,(\int_0^t\left[\,r(s)-\frac{1}{2}\sum_k\int\alpha_k^2(s)\,\right]ds).E(\exp(\sum_{k=1}^{n}\int_0^t \alpha_k(s)\,dB(s))). \tag{12}$$

But, we know if X be a random variable of the Normal distribution with mean μ and variance σ^2 ($X\rightarrow N(\mu,\sigma^2)$) then the moment generating function that is:

$$M_X(u)=\exp(\,\mu.u+\frac{\sigma^2u^2}{2}).$$

Let $X(t)=\int_0^t \alpha(s)\,dB(s)$, then $X(t)\rightarrow N(0,\int_0^t\alpha^2(s)\,ds)$ (see[1]).

Therefore,

$$M_{X(t)}(u)=E(\exp(u\,X(t)))=\exp\,(\frac{u^2}{2}\int_0^t\alpha^2(s)\,ds), \tag{13}$$

With suppose $u=1$, we have:

$$E\,(\exp(X(t)))=E(\exp\,(\sum_{k=1}^{n}\int_0^t\alpha_k(s)\,dB(s)))=\exp(\frac{1}{2}\sum_{k=1}^{n}\int_0^t\alpha^2(s)\,ds) \tag{14}$$

Finally, by substituting relation (14) in (12) we get:

$$E(\,N(t))=E(\,N_0)\exp(\int_0^t r(s)\,ds).$$

Furthermore if $u=2$ in (13),

$$E(\exp(2\sum_{k=1}^{n}\int_0^t \alpha_k(s)dB(s))) = \exp(2\sum_{k=1}^{n}\int_0^t \alpha^2(s)\,ds).$$

So,

$$E(N^2(t)) = E({N_0}^2)\exp(2\int_0^t (r(s)+\frac{1}{2}\sum_k \alpha^2(s))\,ds),$$

$$Var(N(t)) = E(N^2(t)) - E^2(N(t))$$

$$Var(N(t)) = \exp(2\int_0^t r(s)\,ds)\left\{ E(N_0^2)\exp(2\sum_{k=1}^{n}\int_0^t \alpha^2(s)\,ds) - E^2(N_0)\right\}.$$

$$Var(N(t)) = \exp(2\int_0^t r(s)\,ds)\left\{ (Var(N_0)+E^2(N_0))\exp(\sum_{k=1}^{n}\int_0^t \alpha^2(s)\,ds) - E^2(N_0)\right\}.$$

Corollary 1. If N_0 be nonrandom, and $r(t) = r,\ \alpha_k(t) = \alpha\ (k = 1, 2, \cdots, n)$, then :

$$E(N(t)) = N_0\exp(rt)$$
$$Var(N(t)) = N_0^2\exp(2rt)(\exp(\sum \alpha^2 t) - 1)$$

IV. CONFIDENCE INTERVAL

Since *N*(*t*) is a random process, we can construct an confidence interval for it.

Theorem 4.1 Let $\boldsymbol{\alpha}$(t) be non-random such that

$$\int_0^t \alpha^2(s)ds < \infty$$

then (1-$\boldsymbol{\epsilon}$)% confidence interval for N(t) is given by :

$$N(t) = N_0.\exp(\int_0^t \left[r(s) - \frac{1}{2}\sum_k \int \alpha_k^2(s) \right] ds \exp(\pm Z_{\frac{\varepsilon}{2}} \sqrt{\sum_{k=1}^{n}\int_0^t \alpha_k^2(s)ds})$$

proof:

It is easy to see that if (*t*) is non-random such that

$$\int_0^t \alpha^2(s)ds < \infty$$

then its Ito integral $Y(t) = \sum_{k=1}^{n}\int_0^t \alpha_k(s)\,dB(s))$ is a Gaussian process with zero mean and variance given by

$\sum_{k=1}^{n}\int_0^t \alpha_k^2(s)\,ds$. So we can rewrite (9) as

$$N(t) = N_0.\exp(\int_0^t \left[r(s) - \frac{1}{2}\sum_k \int \alpha_k^2(s) \right] ds)\exp(\sum_{k=1}^{n}\int_0^t \alpha_k(s)\,dB(s)).$$

$$N(t) = D(t)\ \exp(\sum_{k=1}^{n}\int_0^t \alpha_k(s)\,dB(s)).$$

where

$$D(t) = N_0 . \exp\left(\int_0^t \left[r(s) - \frac{1}{2}\sum_k \int \alpha_k^2(s) \right] ds\right)$$

Thus

$$\sum_{k=1}^{n} \int_0^t \alpha_k(s)\, dB(s) = \ln \frac{N(t)}{D(t)} \to N(0, \sum_{k=1}^{n}\int_0^t \alpha_k^2(s)\, ds)$$

so we can put

$$-Z_{\frac{\varepsilon}{2}} \sqrt{\sum_{k=1}^{n}\int_0^t \alpha_k^2(s) ds} \le \ln \frac{N(t)}{D(t)} \le Z_{\frac{\varepsilon}{2}} \sqrt{\sum_{k=1}^{n}\int_0^t \alpha_k^2(s)\, ds}$$

$$D(t)\exp(-Z_{\frac{\varepsilon}{2}} \sqrt{\sum_{k=1}^{n}\int_0^t \alpha_k^2(s) ds}) \le N(t) \le D(t)\exp(Z_{\frac{\varepsilon}{2}} \sqrt{\sum_{k=1}^{n}\int_0^t \alpha_k^2(s)\, ds}).$$

V.NUMERICAL ILLUSTRATION

In this section we obtain the output of exponential model and stochastic exponential model with the mixture noise of Iran population in the period 2005-2014. Also, we present a model for estimating of population.

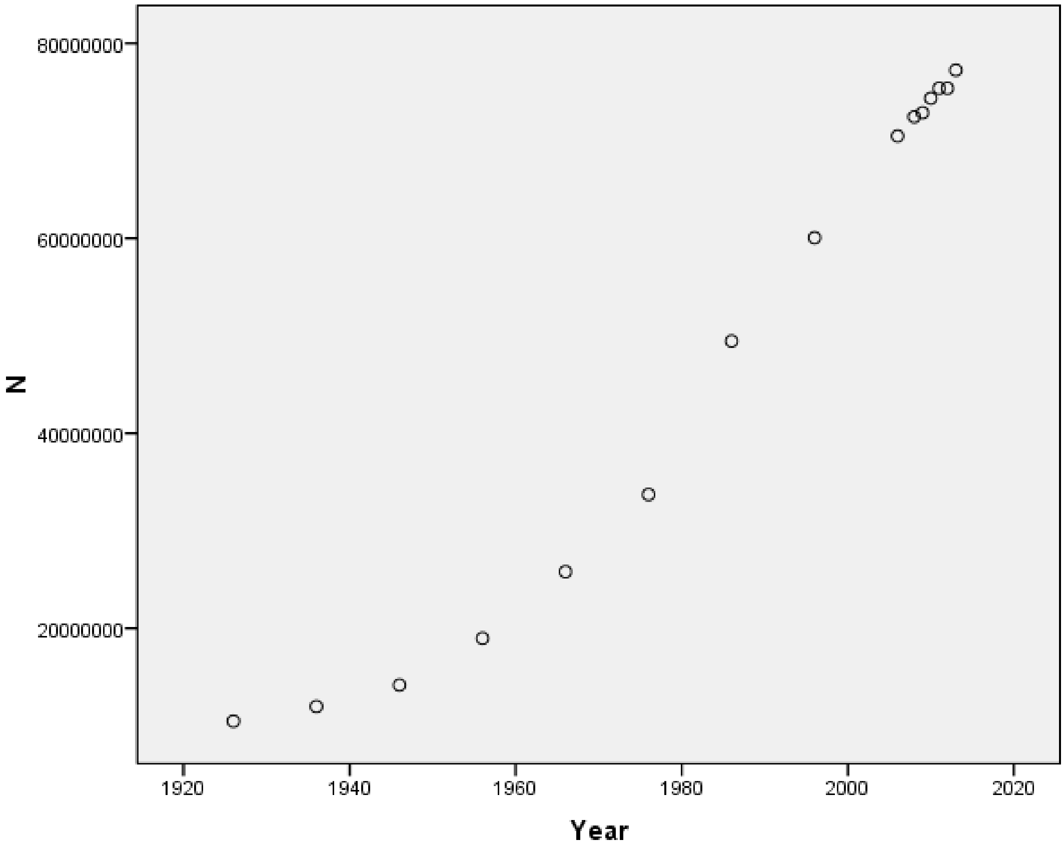
In table 1, we have real number of population. We assume that $t = 0$ correspond to 2005, and so the initial population is $N_0 = 69354000$. The $r(t)$ of this table is obtained by $r(t) = \sqrt[t]{\frac{N(t)}{N_o}} - 1.$

Year	N(t)	r(t)
2005	69354000	-
2006	70766000	0.0101
2007	71590000	0.0106
2008	72464000	0.0110
2009	73388000	0.0113
2010	74363000	0.0116
2011	75388000	0.0119
2012	76340000	0.0120
2013	77254000	0.0121
2014	78141000	0.0120
2015	79264139	0.0122
2016	79926270	0.0118
2017	80876275	-0.0288
2018	83992949	0.0137
2019	84395000	0.0131

Table 1. Population and population growth rate(Iran).

The 2, 3 and 4 columns in the table 1 are the related to the Iran population, Iran population growth rate and prediction of the Iran population via exponential population growth of the period 2005-2019.

In table 2, sdse is standard deviation of stochastic exponential population growth model solution with mixture noise and $\alpha(t)$ indicate the infirmity and intensity of noise at time t.



ANOVA[b]

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	9.915E15	1	9.915E15	351.190	.000[a]
	Residual	3.670E14	13	2.823E13		
	Total	1.028E16	14			

a. Predictors: (Constant), Year

b. Dependent Variable: N

Coefficients[a]

Model		Unstandardized Coefficients		Standardized Coefficients	t	Sig.
		B	Std. Error	Beta		
1	(Constant)	-1.673E9	9.194E7		-18.199	.000
	Year	868439.651	46341.322	.982	18.740	.000

a. Dependent Variable: N

Model Summary and Parameter Estimates

Dependent Variable:N,
Independent variable:Year.

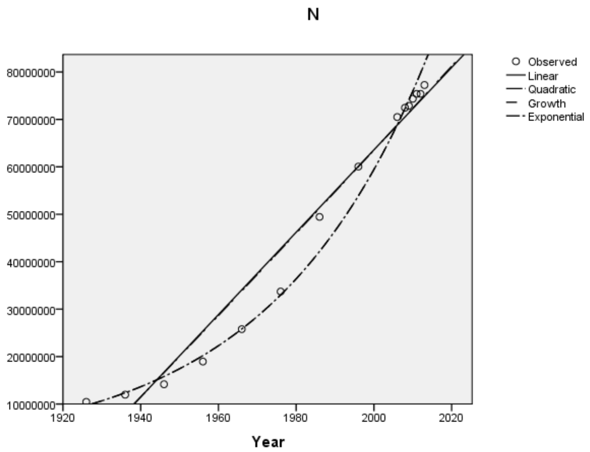
Equation	Model Summary					Parameter Estimates		
	R Square	F	df1	df2	Sig.	Constant	b1	b2

Linear	.964	351.190	1	13	.000	-1.673E9	8.684E5	220.167
Quadratic	.966	370.963	1	13	.000	-8.171E8	.000	
Growth	.991	1.478E3	1	13	.000	-31.161	.025	
Exponential	.991	1.478E3	1	13	.000	2.932E-14	.025	

As can be seen, for each of these models, values (determination coefficient), computational F, degree of error freedom, significance level F and estimation of model parameters are obtained. Now we choose the model with the determination coefficient maximum value. Here we select the view model, the general form of which is as follows:

$$y_i = \beta_0 e^{\beta_1 x_i}$$

Therefore, the population of Iran in 2020 is estimated at 86843000.



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