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A Note on Fractional Exponential Function

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ABSTRACT:

In this paper, we study the fractional exponential function.Some properties of fractional exponential function are obtained. A new multiplication of fractional analytic functions plays an important role in this article. In fact, our results are generalizations of the properties of classical exponential function.

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I. INTRODUCTION

In recent years, the application of fractional calculus in many different fields, such as physics, control, mechanics, dynamics, bioengineering, economics, electrical engineering, viscoelasticity, signal processing, has aroused people's strong interest [1-6].At present, the definitions of fractional calculus mainly include Riemann-Liouville (R-L) type, Caputo type, Grunwald-Letnikov (G-L) type, Weyl type, Riesz type, Jumarie type, and so on [7-11].

This paper studies the fractional exponential function. Some properties of fractional exponential function are obtained. On the other hand, a new multiplication of fractional analytic functions plays an important role in this article. In fact, our results are generalizations of the properties of classical exponential function.

II. PRELIMINARIES

Firstly, the definition of fractional analytic function is introduced.

Definition 2.1([12])**:** Suppose that x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \le 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. On the other hand, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.2([13])**:** Assume that $0 < \alpha \le 1$, and x_0 is a real number. Suppose that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \quad (1)$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \quad (2)$$

Then

$$= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)(x-x_0)^{k\alpha}. \quad (3)$$

Equivalently,

$f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m \right) \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^{\alpha} \right)^{\otimes k}. \quad (4)$$

Definition 2.3 ([14])**:** Suppose that$0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (x - x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^{\alpha} \right)^{\otimes k}, \quad (5)$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} (x - x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^{\alpha} \right)^{\otimes k}. \quad (6)$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(g_\alpha(x^\alpha) \right)^{\otimes k}, \quad (7)$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(f_\alpha(x^\alpha) \right)^{\otimes k}. \quad (8)$$

Definition 2.4 ([14])**:** Let$0 < \alpha \leq 1$. If $f_\alpha(x^\alpha)$,$g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha. \quad (9)$$

Then $f_\alpha(x^\alpha)$,$g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.5([14])**:** Suppose that$0 < \alpha \leq 1$, and x isa real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes k}. \quad (10)$$

And$Ln_\alpha(x^\alpha)$is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes 2k}, \quad (11)$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes (2k+1)}. \quad (12)$$

Definition 2.6([15])**:** Assume that n is a positive integer, if $f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes -1}$.In addition, $\left(f_\alpha(x^\alpha)\right)^{\otimes n} = f_\alpha(x^\alpha) \otimes \cdots \otimes f_\alpha(x^\alpha)$ is called the$n$th power of $f_\alpha(x^\alpha)$. And $\left(f_\alpha(x^\alpha)\right)^{\otimes -n} = \left(f_\alpha(x^\alpha)\right)^{\otimes -1} \otimes \cdots \otimes \left(f_\alpha(x^\alpha)\right)^{\otimes -1}$ is the$n$th power of $\left(f_\alpha(x^\alpha)\right)^{\otimes -1}$. Moreover, we define$\left(f_\alpha(x^\alpha)\right)^{\otimes 0} = 1$.

Definition 2.7 ([16])**:** Assume that$0 < \alpha \leq 1$. Let$f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ beα-fractional analytic functions. Then theα-fractional analytic function$f_\alpha(x^\alpha)^{\otimes g_\alpha(x^\alpha)}$ is defined by

$$f_\alpha(x^\alpha)^{\otimes g_\alpha(x^\alpha)} = E_\alpha\left(g_\alpha(x^\alpha) \otimes Ln_\alpha\left(f_\alpha(x^\alpha)\right) \right). \quad (13)$$

III. MAIN RESULTS

In this section, the main results in this paper are introduced.

Theorem 3.1:*Let$0 < \alpha \leq 1$, then the α-fractional exponential function*

$$E_\alpha(x^\alpha) = e_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)} x^\alpha}, \quad (14)$$

where$e_\alpha = E_\alpha(1) = \sum_{k=0}^{\infty} \frac{[\Gamma(\alpha+1)]^k}{\Gamma(k\alpha+1)}$.

Proof By Definition 2.7,

$e_\alpha^{\otimes \frac{1}{\Gamma(\alpha+1)} x^\alpha}$

$$= E_\alpha\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes Ln_\alpha(e_\alpha)\right)$$

$$= E_\alpha\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes Ln_\alpha\big(E_\alpha(1)\big)\right)$$

$$= E_\alpha\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes 1\right)$$

$$= E_\alpha(x^\alpha).$$

On the other hand, if $\frac{1}{\Gamma(\alpha+1)} x^\alpha = 1$, then $x^\alpha = \Gamma(\alpha+1)$, and hence $x^{k\alpha} = [\Gamma(\alpha+1)]^k$. Therefore, by Definition 2.5,

$$e_\alpha = E_\alpha(1) = \sum_{k=0}^{\infty} \frac{[\Gamma(\alpha+1)]^k}{\Gamma(k\alpha+1)}.$$

Q.e.d.

Remark 3.2: If $\alpha = 1$, then

$$e_1 = E_1(1) = \sum_{k=0}^{\infty} \frac{1}{k!} = e. \qquad (15)$$

And

$e_1^{\otimes x} = E_1(x) = e^x$, (16)

which is the classical exponential function.

Theorem 3.3: *If* $0 < \alpha \le 1$, *then the* α*-fractional exponential function*

$$E_\alpha(x^\alpha) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(n)_k}{n^k} \cdot \frac{1}{\Gamma(k\alpha+1)} x^{k\alpha}. \qquad (17)$$

And the α*-fractional exponentialconstant*

$$e_\alpha = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(n)_k}{n^k} \cdot \frac{[\Gamma(\alpha+1)]^k}{\Gamma(k\alpha+1)}. \qquad (18)$$

Where $(n)_k = n(n-1)\cdots(n-k+1)$ *and* $(n)_0 = 1$.

Proof $E_\alpha(x^\alpha)$

$$= \lim_{n\to\infty}\left(1 + \frac{1}{n} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes n}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k} \cdot \left(\frac{1}{n} \cdot \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \cdot \frac{1}{n^k}\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \cdot \frac{1}{n^k} \cdot \frac{k!}{\Gamma(k\alpha+1)} x^{k\alpha}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{n!}{(n-k)!} \cdot \frac{1}{n^k} \cdot \frac{1}{\Gamma(k\alpha+1)} x^{k\alpha}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(n)_k}{n^k} \cdot \frac{1}{\Gamma(k\alpha+1)} x^{k\alpha}.$$

Moreover,

$$e_\alpha = E_\alpha(1) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(n)_k}{n^k} \cdot \frac{[\Gamma(\alpha+1)]^k}{\Gamma(k\alpha+1)}.$$

Q.e.d

Corollary 3.4:*Let* $0 < \alpha \leq 1$, *then*

$$\sum_{k=0}^{\infty} \frac{[\Gamma(\alpha+1)]^k}{\Gamma(k\alpha+1)} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(n)_k}{n^k} \cdot \frac{[\Gamma(\alpha+1)]^k}{\Gamma(k\alpha+1)}. \quad (19)$$

Remark 3.5: If $\alpha = 1$, then by Corollary 3.4, we obtain

$$\sum_{k=0}^{\infty} \frac{1}{k!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(n)_k}{n^k} \cdot \frac{1}{k!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n. \quad (20)$$

IV. CONCLUSION

In this paper, we discuss some properties of fractional exponential function. These properties are generalizations of the properties of traditional exponential function.A new multiplication of fractional analytic functions plays an important role in this article.In the future, we will use fractional exponential function to solve the problems in fractional calculus and fractional differential equations.

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