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Triangular Graphs and the Pell Equation

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Abstract

A nontrivial graph is *triangular* if **(1)** its vertices are labeled by distinct triangular numbers, **(2)** each edge is weighted by the product of the labels of its end vertices, and **(3)** each edge weight is a distinct triangular number. Various theorems are presented.

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I. Introduction

The n-th triangular number, t_n, is given by

$$t_n = \frac{n(n+1)}{2} = 1 + 2 + 3 + ... + n \qquad (1)$$

where n is a natural number. A nontrivial graph is triangular if **(1)** its vertices are labeled by distinct triangular numbers, **(2)** each edge is weighted by the product of the labels of its end vertices, and **(3)** each edge weight is a distinct triangular number.

Similar graphs in which the vertex and edge labels are oblong, (of the form $n(n + 1)$), and k-long (of the form $n(n + k)$, where k is a fixed positive integer) have been studied. [4, 5, 6]. Using Pell equations, we shall obtain infinite classes of triangular graphs.

There are infinitely many triangular numbers, such as $t_8 = 36$, that are squares, but they thin out rapidly as n goes to infinity. Also, a given number m is triangular if and only if $1 + 8m$ is a perfect square. Finally, we shall call t_n and t_m *compatible*, if the product t_nt_m is triangular. For example, $t_2 = 3$ and $t_5 = 15$ are compatible since $t_2t_5 = 3\times15 = 45 = t_9$.

II. *k*-long Graphs

In [3], a number is called *k-long* if it is of the form $n(n + k)$. This concept generalizes oblong numbers, that is, numbers of the form $n(n + 1)$. It is shown there that the product of two consecutive k-long numbers is k-long. In [4, 5], a graph G is called *k-long* if

1. The vertices of G can be labeled with distinct k-long numbers.
2. The weights of the edges are the products of their endvertices.
3. The weights are distinct k-long numbers.

It should be noted that the set of labels and the set of weights need not be disjoint.

III. Triangular Graphs

In the spirit of Section II, a *triangular graph* is defined as follows. A graph, G, is triangular if

1. The vertices of G can be labeled with distinct triangular numbers.
2. The weights of the edges are the products of the labels of their endvertices.
3. The weights are distinct triangular numbers.

As was the case with k-long graphs, the set of labels and the set of weights need not be disjoint.

We suspect that the product of any two consecutive triangular numbers (not including 1) is never triangular, thereby making triangular graphs more difficult to construct. A computer search involving the first ten million triangular numbers strengthened our suspicion.

The path P_5 is triangular, as can be seen by labeling the vertices sequentially 1, 3, 15, 66, and 406. These labels generate the weights (sequentially) 3, 45, 990, and 26796. The vertex labels are t_1, t_2, t_5, t_{11}, and t_{28}, and the edge weights are t_2, t_9, t_{44}, and t_{231}.We shall see shortly, that all paths are triangular

IV. The Pell Equation and its Generalization

The Pell equation $x^2 - ky^2 = 1$ is solvable in positive integers provided that k is not a square [7]. It will be useful to convert the solution $x = a$ and $y = b$ into the formal expression $a+b\sqrt{k}$. If b is the smallest (positive) solution, then all solutions are the coefficients of $\left(a+b\sqrt{k}\right)^n$.

The generalized Pell equation $x^2 - ky^2 = j$ may or may not be solvable. For example, $x^2 - 3y^2 = 2$ has no solution in integers as can be seen by writing both sides mod 3, yielding $x^2 = 2 \pmod 3$, which has no solution. On the other hand, if the generalized Pell equation $x^2 - ky^2 = j$ has a solution, $x = c$ and $y = d$, then it has the infinitely many solutions expressed formally as $\left(c+d\sqrt{k}\right)\left(a+b\sqrt{k}\right)^n$. See [7]

Example: Given the generalized Pell equation $x^2 - 3y^2 = 4$, with solution $x = 4$, $y = 2$, first solve the associated ordinary Pell equation $x^2 - 3y^2 = 1$ and obtain its formally expressed solutions $\left(2+\sqrt{3}\right)^n$. Then the given generalized Pell equation has the formal solutions

$$\left(4+2\sqrt{3}\right)\left(2+\sqrt{3}\right)^n$$

$n = 2$, for example, yields $\left(4+2\sqrt{3}\right)\left(2+\sqrt{3}\right)^2 = \left(4+2\sqrt{3}\right)\left(7+4\sqrt{3}\right)=52+30\sqrt{3}$, from which one obtains the solution $x = 52$ and $y = 30$.

V. Finding Compatibles

The Pell equations will now be employed to find infinitely many compatibles, t_r, for a given non-square triangular number k. That is, indices, r, must be found for which kt_r is triangular. (Note that triangular numbers that are squares can easily be avoided in the construction of triangular graphs, as they have zero density in the set of triangular numbers.)

Using the fact that a given number, m, is triangular if and only if $1 + 8m$ is a perfect square, the following sequence of equations are obtained.

$$8kt_r + 1 = s^2$$
$$8k\frac{r(r+1)}{2} + 1 = s^2$$
$$4kr(r+1) + 1 = s^2$$
$$k(4r^2+4r+1) + (1-k) = s^2$$
$$k(2r+1)^2 + (1-k) = s^2$$
$$s^2 - k(2r+1)^2 = 1-k$$

Letting $x = 2r+1$, results in the generalized Pell equation $s^2 - kx^2 = 1-k$, which can be written as

$$\bar{s}^2 - k\bar{x}^2 = 1-k \tag{3}$$

to distinguish its solutions from those of the associated Pell equation $s^2 - kx^2 = 1$.

Since $\bar{x} = 2r+1$, it must be odd in order to obtain integer values of r. Observe that (3) has the formally expressed solution $1+\sqrt{k}$, since $\bar{s}_0 = \bar{x}_0 = 1$ is a solution.

Two lemmas are required.

Let $s_1 + x_1\sqrt{k}$ be a formal solution to the Pell equation, $s^2 - kx^2 = 1$. Then the n-th solution, $s_n + x_n\sqrt{k} = (s_1 + x_1\sqrt{k})^n$. It follows that

$$s_{n+1} + x_{n+1}\sqrt{k} = (s_n + x_n\sqrt{k})(s_1 + x_1\sqrt{k})$$

Equating coefficients of 1 and $\sqrt{k}$ yields Lemma 1.

Lemma 1:

$$s_{n+1} = s_n s_1 + k x_n x_1$$
$$x_{n+1} = s_n x_1 + x_n s_1$$

The n-th formal solution to (3), $\bar{s}_n + \bar{x}_n\sqrt{k} = (s_n + x_n\sqrt{k})(1+\sqrt{k})$. Equating coefficients yields:

Lemma 2:

$$\bar{s}_n = s_n + kx_n$$
$$\bar{x}_n = s_n + x_n$$

It has now been established that the generalized Pell equation, $\bar{s}^2 - k\bar{x}^2 = 1 - k$ has infinitely many solutions. Recall, however, that $\bar{x}$ must be odd, since $\bar{x} = 2r + 1$, where *r* is the index of a triangular number compatible with k. In light of the second equation of Lemma 2, this translates into the requirement that s_n and x_n have opposite parity. To show this, two cases must be considered.

Case 1: *k* is odd.

It can be seen from the Pell equation $s^2 - kx^2 = 1$, that x^2 and s^2 have opposite parity, from which it follows that *s* and *x* have opposite parity, and we are done.

Case 2: *k* is even.

It can be seen from the Pell equation that $s_n$isalways odd. Then the second equation of Lemma 1 implies that $x_{n+1} \equiv x_1 + x_n$(mod 2). If x_1 is even, this becomes $x_{n+1} \equiv x_n$(mod 2), implying that x_n is always even, and we are done. If x_1 is odd,then $x_{n+1} \equiv 1 + x_n$(mod 2), so the parity of x_n will alternate. This will yield infinitely many solutions in which s_n and x_n have opposite parity, and the lemma is proven.

In summary, every non-square triangular number has infinitely many compatibles. Infinite classes of triangular graphs will now be obtained.

VI. Infinite Classes of Triangular Graphs

Theorem 1: All paths are triangular.

Proof: Given *n*, let S_n be a strictly increasing sequence of *n* triangular numbers such that any two consecutive members of S_n are compatible. Such a sequence exists, for all *n*, by the result of the previous section. Then the path P_n is indeed triangular. Simply label its vertices consecutively using the members of S_n. □

Theorem 2: All cycles are triangular.

Proof: Given *n*, let S_{n-1}= (s_1, s_2, s_3, …, s_{n-1}), be a strictly increasing sequence of $n-1$ triangular numbers such that any two consecutive members of S_{n-1} are compatible, and $s_1 = 1$. Now given the cycle, C_n, label any $n - 1$ consecutive vertices with the triangular numbers of S_{n-1}. Since these triangular numbers are strictly increasing as we go around the cycle, we will obtain a strictly increasing sequence of weights, w_1, w_2, …, w_{n-2}.

Denote the label of the final vertex by *t*. Choose *t* so that it is compatible with, and greater than, s_{n-1}. (Note that any *t* is compatible with 1.) Then the penultimate weight, w_{n-1}, is strictly greater than w_{n-2} since $t > s_{n-1}$. It follows that the weights, w_1, w_2, …, w_{n-1} are distinct since they form a strictly increasing sequence. Unfortunately, the final weight, $w_n = t < w_{n-1}$, since $w_{n-1} = ts_{n-1}$. To ensure that w_n does not equal any of the previous weights w_1, w_2, …, w_{n-2}, increase *t* if necessary so that it is greater than w_{n-2}, which can be done since s_{n-1} has infinitely many compatibles. □

Theorem 3: All trees are triangular.

Proof: Let T be a rooted tree. Label the root with a triangular number which is denoted by a_{11}. Label the second row of vertices from left to right by the strictly increasing triangular numbers a_{21}, a_{22}, a_{23}… where all of these labels are compatible with a_{11}. If we denote the weights of the edges linking the first and second rows of vertices (from left to right) by w_{11}, w_{12}, w_{13}…, it follows that these weights form a strictly increasing sequence.

Choose the triangular number, a_{31}, that is the first vertex of the third row, so that it is compatible with its parent vertex label, and is greater than the maximum of the weights of all previous edges. This will ensure that the edge weights are distinct. As was the case with the second row of vertices, the sequence of labels of the vertices of the third row, a_{31}, a_{32}, a_{33}, is strictly increasing. This guarantees that the vertex labels and edge labels thus far are distinct. Since each triangular number (except squares which are not considered) has infinitely many compatibles, arbitrarily large trees can be accommodated by continuing this algorithm.

The algorithm terminates when the desired labeling is achieved. □

Theorem 4: All unicyclic graphs are triangular.

Proof: Label the cycle using the algorithm of Theorem 2. Let *j* be the smallest index for which deg (a_j) > 2. Treating a_j as the root of the pendant tree attached to it, label the first vertex of the second row of its pendant tree so that it is strictly greater than all the weights thus far. The remainder of the labeling using the algorithm of Theorem 3 will ensure the distinctness of the vertex labels and edge weights thus far. Note that a_j is the initial vertex in the subsequence of vertices with degrees greater than 2. Continue the algorithm until the pendant trees of these vertices are fully labeled. □

Theorem 5: All bicyclic graphs such that the two cycles share exactly one vertex are triangular.

Proof: Label the shared vertex by 1, and apply several of the above algorithms. □

Corollary 1: Let graph G contain an arbitrary number of cycles all containing a vertex, *v*, and such that the intersection of the vertices of any pair of cycles is *v*. G may contain tree structures. Then G is triangular.

Proof: Assign the label 1 to *v*. Then apply the algorithms of Theorems 3 and 4. □

Theorem 6: Let graph G be constructed as follows. Begin with the cycle C_n and label one of its vertices, *v*, with 1. Then add $n - 3$ edges so that deg(*v*) = $n - 1$. Then G is triangular.

Proof: Label the vertices of the initial C_n using the algorithm of Theorem 2. Note that the added edges of the second step of the construction have the same weights as the vertices they are attached to that are not labeled 1. Hence, they will all be distinct. □

Remark 1: Note that the graph of Theorem 6 is *pancyclic*, that is, G contains each of the cycles C_3, C_4, …, C_n. See [8, 9]. On the other hand, if tree structure is added to G, it is still triangular but is no longer pancyclic, since it is no longer *hamiltonian* (that is, it has no spanning cycle).

VII. Conclusion

We close with the following definition and open questions.

Definition: The distinct triangular numbers t_n and t_m, are *mutually compatible*, if there exists a triangular number other than 1 that is compatible with both of them.

Example: $t_2 = 3$ and $t_{11} = 66$ are mutually compatible, since they are both compatible with $t_5 = 15$.

Question 1: Are any given pair of distinct triangular numbers compatible? A positive answer would dramatically enhance the discovery of new classes of triangular graphs, for example, the wheel, W_n.

Question 2: As was previously indicated, with the exception of 1 and 3,we have not found a pair of consecutive compatible triangular numbers among the first ten million triangular numbers. We conjecture that there are no such pairs, and seek a proof or counterexample.

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