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On Homogeneous Bi-quadratic Diophantine Equation with Five Unknowns

$$x^4 - y^4 = 5^{2n}\left(z^2 - w^2\right)T^2$$

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Abstract:

The Bi-quadratic Diophantine equation with five unknowns given by $x^4 - y^4 = 5^{2n}\left(z^2 - w^2\right)T^2$ is analyzed for its non-zero distinct integer solutions . A few interesting relations among the solutions are presented.

Keywords: Bi-quadratic equation with five unknowns, Homogeneous bi-quadratic, Integral solutions.

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1.Introduction:

The Diophantine equations are rich in variety and offer an unlimited field for research [1-3]. In particular refer [4-13] for a few problems on Biquadratic equation with 2, 3,4 and 5 unknowns. This paper concerns with yet another interesting Biquadratic Diophantine equation with five variables given by $x^4 - y^4 = 5^{2n}\left(z^2 - w^2\right)T^2$ for determining its infinitely many non-zero distinct integral solutions. A few interesting relations among the solutions are presented.

2.Method of Analysis:

The homogeneous Bi-quadratic diophantine equation with five variables under consideration is

$$x^4 - y^4 = 5^{2n}\left(z^2 - w^2\right)T^2 \qquad (1)$$

Method 1:
Introducing the linear transformations

$$x = 5^n(u+v), y = 5^n(u-v), T = 5^n P, z = 2u+v, w = 2u-v, u \neq v \qquad (2)$$

in (1), it reduces to the Pythagorean equation

$$u^2 + v^2 = P^2 \qquad (3)$$

whose solutions may be taken as

$$v = 2ab, u = a^2 - b^2, P = a^2 + b^2, a \geq b \geq 0 \qquad (4)$$

In view of (2), the corresponding integer solutions to (1) are given by

$$\left.\begin{aligned} x &= x(a,b) = 5^n(a^2 - b^2 + 2ab) \\ y &= y(a,b) = 5^n(a^2 - b^2 - 2ab) \\ z &= z(a,b) = 2(a^2 - b^2 + ab) \\ w &= w(a,b) = 2(a^2 - b^2 - ab) \\ T &= T(a,b) = 5^n(a^2 + b^2) \end{aligned}\right\} \quad (5).$$

Relations among the solutions:

1. $xy + T^2$ is twice a perfect square

2. Each of the following expressions is a square multiple of 5^n

$T(a,b) \pm (2x(a,b) - 5^n z(a,b))$,

$T(a,b) \pm (5^n w(a,b) - 2y(a,b))$,

$2(x(q^2, p^2 - q^2) + T(q^2, p^2 - q^2))$, $p \geq q \geq 0$

$2(y(p^2, p^2 - q^2) + T(p^2, p^2 - q^2))$

3.Each of the following expressions is a cube multiple of 5^n

$2(x(p, 2p^2 - p) + T(p, 2p^2 - p))$,

$2(y(p^2, p^2 - q^2) + T(p^2, p^2 - q^2))$

4.Each of the following expressions is a nasty number:

$30(T^2(a,b) + 5^{2n} z(a,b) w(a,b))$,

$5^n(y(a,b) + 3T(a,b) + 5^n z(a,b))$

Note:1

Apart from (2),one may consider the following transformations

$x = 5^n(u + v), y = 5^n(u - v), T = 5^n P, z = u + 2v, w = u - 2v, u \neq v$

$x = 5^n(u + v), y = 5^n(u - v), T = 5^n P, z = 2uv + 1, w = 2uv - 1, u \neq v$

leading to two different solutions to (1).

Method 2:

Introducing the linear transformations

$$x = 5^n(2p), y = 5^n(2q), T = 5^n(2Q), z = 6p + 4q, w = 4p + 6q \quad (6)$$

in (1) ,it reduces to

$$p^2 + q^2 = 5Q^2 \quad (7)$$

Assume

$$Q = a^2 + b^2 \quad (8)$$

Write 5 as

$$5 = (2 + i) * (2 - i) \quad (9)$$

Substituting (8) & (9) in (7) and employing the method of factorization,consider

$$p + iq = (2 + i)(a + ib)^2 \quad (10)$$

from which we have

$$\left.\begin{aligned} p &= 2(a^2 - b^2 - 2ab), \\ q &= a^2 - b^2 + 4ab \end{aligned}\right\} \qquad (11)$$

Substituting (8) & (11) in (6),the corresponding integer solutions to (1) are given by

$$x = 5^n(4a^2 - 4b^2 - 8ab),$$

$$y = 5^n(2a^2 - 2b^2 + 8ab),$$

$$T = 5^n(2a^2 + 2b^2),$$

$$z = (16a^2 - 16b^2 - 8ab),$$

$$w = (14a^2 - 14b^2 + 8ab)$$

Note :2

The integer 5 on the R.H.S. of (7) is also represented as below:

$$5 = (1 + 2i) * (1 - 2i),$$

$$5 = (\sqrt{5}i) * (-\sqrt{5}i),$$

$$5 = \frac{(11 + 2i) * (11 - 2i)}{25},$$

$$5 = \frac{(2 + 11i) * (2 - 11i)}{25},$$

$$5 = \frac{(2 + 29i) * (2 - 29i)}{169},$$

$$5 = \frac{(29 + 2i) * (29 - 2i)}{169}$$

Repeating the above process,one obtains six more different solutions to (1).

Method 3:

Write (7) as

$$5Q^2 - q^2 = p^2 = p^2 * 1 \qquad (12)$$

Assume

$$p = 5a^2 - b^2 \qquad (13)$$

Write 1 as

$$1 = (\sqrt{5} + 2) * (\sqrt{5} - 2) \qquad (14)$$

Substituting (13) & (14) in (12) and employing the method of factorization ,consider

$$\sqrt{5}Q + q = (\sqrt{5} + 2)(\sqrt{5}a + b)^2 \qquad (15)$$

from which we have

$$\left.\begin{aligned} Q &= 5a^2 + b^2 + 4ab, \\ q &= 2(5a^2 + b^2 + 5ab) \end{aligned}\right\} \quad (16)$$

Substituting (13) & (16) in (6),the corresponding integer solutions to (1) are given by

$$x = 2 * 5^n * (5a^2 - b^2),$$

$$y = 4 * 5^n * (5a^2 + b^2 + 5ab),$$

$$T = 2 * 5^n * (5a^2 + b^2 + 4ab),$$

$$z = (70a^2 + 2b^2 + 40ab)$$

$$w = (80a^2 + 8b^2 + 60ab)$$

Note :3

The integer 1 on the R.H.S. of (12) is also represented as below:

$$1 = \frac{(\sqrt{5}+1)*(\sqrt{5}-1)}{4},$$

$$1 = \frac{(5\sqrt{5}+2)*(5\sqrt{5}-2)}{121}, 1 = \frac{(5\sqrt{5}+11)*(5\sqrt{5}-11)}{4},$$

$$1 = \frac{(13\sqrt{5}+2)*(13\sqrt{5}-2)}{841}, 1 = \frac{(13\sqrt{5}+19)*(13\sqrt{5}-19)}{484},$$

$$1 = \frac{(13\sqrt{5}+29)*(13\sqrt{5}-29)}{4}, 1 = \frac{(17\sqrt{5}+1)*(17\sqrt{5}-1)}{1444},$$

$$1 = \frac{(17\sqrt{5}+22)*(17\sqrt{5}-22)}{961}, 1 = \frac{(17\sqrt{5}+31)*(17\sqrt{5}-31)}{484},$$

$$1 = (17\sqrt{5}+38)*(17\sqrt{5}-38)$$

Repeating the above process,one obtains ten sets of different solutions to (1).

Method 4:

Introduction of the transformations

$$p = 2R, q = X + 5S, Q = X + S \quad (17)$$

in (7) leads to

$$X^2 = R^2 + 5S^2 \quad (18)$$

which is satisfied by

$$S = 2ab, R = 5a^2 - b^2, X = 5a^2 + b^2 \quad (19)$$

Substituting (19) in (17) and using (6),the corresponding integer solutions to (1) are given by

$$x = 2*5^n*(10a^2 - 2b^2),$$

$$y = 2*5^n*(5a^2 + b^2 + 10ab),$$

$$T = 2*5^n*(5a^2 + b^2 + 2ab),$$

$$z = (80a^2 - 8b^2 + 40ab)$$

$$w = (70a^2 - 2b^2 + 60ab)$$

Note:4

Write (18) as a system of double equations as in Table:1 below:

Table:1-system of double equations

system	I	II
$X + R$ for	$5S^2$	S^2
$X - R$	1	5

Solve each of the above two systems for S, X, R .Then, from (17) and (6),one obtains the corresponding integer solutions to (1).For simplicity,the integer solutions to (1) are exhibited
Below:

Solutions from system I:

$$x = 2*5^n*(20k^2 + 20k + 4),$$

$$y = 2*5^n*(10k^2 + 20k + 8),$$

$$T = 2*5^n*(10k^2 + 12k + 4),$$

$$z = (160k^2 + 200k + 56),$$

$$w = (140k^2 + 200k + 64)$$

Solutions from system II:

$$x = 2*5^n*(4k^2 + 4k - 4),$$

$$y = 2*5^n*(2k^2 + 12k + 8),$$

$$T = 2*5^n*(2k^2 + 4k + 4),$$

$$z = (32k^2 + 72k + 8),$$

$$w = (28k^2 + 88k + 32)$$

3.Conclusion:

An attempt has been made to obtain non-zero distinct integer solutions to the homogeneous bi-quadratic diophantine equation with five unknowns given by $x^4 - y^4 = 5^{2n}(z^2 - w^2)T^2$
One may search for other sets of integer solutions to the considered equation as well as other choices of the fourth degree diophantine equations with multivariables.

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