International Journal of Engineering Inventions
e-ISSN: 2278-7461, p-ISSN: 2319-6491
Volume 12, Issue 3 [March. 2023] PP: 249-253

Higher Order Linear Equations with Constant Coefficients

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ABSTRCT;
In this paper, I have solve some problems with Higher Order Linear Equations with Constant Coefficients using some formula's from Higher order differential equation.
i.e, $f(D)y = sinax, f(D)y = cosax, f(D)y = e^{ax}, f(D)y = e^{ax}V, f(D)y = x^k,$

$f(D)y = x^kV$

Keywords :
System of Higher-Order Linear Differential Equations, Operator Problem, Vector Green's Function, Initial Value Problems, $f(D)y = e^{ax}, f(D)y = sinax, f(D)y = e^{ax}V,$
$f(D)y = x^k, f(D)y = x^kV, f(D)y = cosax$

Date of Submission: 11-03-2023 Date of acceptance: 25-03-2023

I. INTRODUCTION

We have just seen that some higher-order differential equations can be solved using methods for first-order equations after applying the substitution v = dy/dx . Unfortunately, this approach has its limitations.

Moreover, as we will later see, many of those differential equations that can be so solved can also be solved much more easily using the theory and methods that will be developed in the next few chapters.

This theory and methodology apply to the class of "linear" differential equations. This is a rather large class that includes a great many differential equations arising in applications. In fact, so important is this class of equations and so extensive is the theory for dealing with these equations, that we will not seriously consider higher-order nonlinear differential equations .

The solutions of linear differential equations with constant coefficients ofthe third order or higher can be found in similar ways as the solutions of second order linear equations. For an n-th order homogeneous linear equation with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0, \qquad a_n \neq 0.$$

It has a general solution of the form

$$y = C_1 y_1 + C_2 y_2 + \dots\dots\dots\dots C_{n-1} y_{n-1} + C_n y_n$$

Where $y_1, y_2, \dots, y_{n-1}, y_n$ are any n linearly independent solutions of the equation. (Thus, they form a set of fundamental solutions of the differentialequation.) The linear independence of those solutions can be determined bytheir Wronskian,

i.e., $\boldsymbol{W}(y_1, y_2, \dots, y_{n-1}, y_n)(t) \neq 0.$

Note 1:
In order to determine the n unknown coefficients C_i, each n-th order equation requires a set of n initial conditions in an initial value problem: $y(t_0) = y_0$, $y'(t_0) = y'_0$, $y''(t_0) = y''_0$, and $y^{(n-1)}(t_0) = y^{(n-1)}_0$.

Note 2:
The Wronskian $\boldsymbol{W}(y_1, y_2, \dots, y_{n-1}, y_n)(t)$ is defined to be thedeterminant of the

$$\begin{bmatrix} y_1 & y_2 & \cdots & \cdots & y_n \\ y'_1 & y'_2 & \cdots & \cdots & y'_n \\ y''_1 & y''_2 & \cdots & \cdots & y''_n \\ \vdots & \vdots & & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & \cdots & y_n^{(n-1)} \end{bmatrix}.$$

Such a set of linearly independent solutions, and therefore, a general solutionof the equation, can be found by first solving the differential equation's characteristic equation:

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_2 r^2 + a_1 r + a_0 = 0.$$

This is a polynomial equation of degree n, therefore, it has n real and or complex roots (not necessarily distinct). Those necessary n linearly independent solutions can then be found using the four rules below.

(i). If r is a distinct real root, then $y = e^{rt}$ is a solution.

(ii). If $r = \lambda \pm \mu i$ are distinct complex conjugate roots, then $y = e^{\lambda t}\cos \mu t$ and $y = e^{\lambda t}\sin \mu t$ are solutions.

(iii). If r is a real root appearing k times, then $y = e^{rt}$, $y = te^{rt}$, $y = t^2 e^{rt}$, … , and $y = t^{k-1} e^{rt}$ are all solutions.

(iv). If $r = \lambda \pm \mu i$ are complex conjugate roots each appears k times, then

$$\begin{array}{ll} y = e^{\lambda t}\cos \mu t, & y = e^{\lambda t}\sin \mu t, \\ y = t e^{\lambda t}\cos \mu t, & y = t e^{\lambda t}\sin \mu t, \\ y = t^2 e^{\lambda t}\cos \mu t, & y = t^2 e^{\lambda t}\sin \mu t, \\ \vdots & \vdots \\ \vdots & \vdots \end{array}$$

$y = t^{k-1} e^{\lambda t} \cos \mu t$, and $y = t^{k-1} e^{\lambda t} \sin \mu t$,

are all solutions.

Example: $y^{(4)} - y = 0$

The characteristic equation is $r^4 - 1 = (r^2 + 1)(r + 1)(r - 1) = 0$,which has roots $r = 1, -1, i, -i$. Hence, the general solution is

$y = C_1e^{t} + C_2e^{-t} + C_3\cos t + C_4\sin t.$

Example:

$$y^{(5)} - 3y^{(4)} + 3y^{(3)} - y'' = 0$$

The characteristic equation is $r^5 - 3r^4 + 3r^3 - r^2 = r^2(r - 1)^3 = 0$, which has roots $r = 0$ (a double root), and 1 (a triple root). Hence, thegeneral solution is

$y = C_1e^{0t} + C_2te^{0t} + C_3e^{t} + C_4te^{t} + C_5t^2e^{t}$
$= C_1 + C_2t + C_3e^{t} + C_4te^{t} + C_5t^2e^{t}$.

Example:

$$y^{(4)} + 4y^{(3)} + 8y'' + 8y' + 4y = 0$$

The characteristic equation is $r^4 + 4r^3 + 8r^2 + 8r + 4 = (r^2 + 2r + 2)^2$
$= 0$, which has roots $r = -1 \pm i$ (repeated). Hence, the general solutionis

$y = C_1e^{-t}\cos t + C_2e^{-t}\sin t + C_3te^{-t}\cos t + C_4te^{-t}\sin t$.

Example:

What is a 4th order homogeneous linear equation whose generalsolution is
$y = C_1e^{t} + C_2e^{2t} + C_3e^{3t} + C_4e^{4t}$?

The solution implies that $r = 1, 2, 3$, and 4 are the four roots of the characteristic equation. Therefore, $r - 1$, $r - 2$, $r - 3$, and $r - 4$ are itsfactors. Consequently, the characteristic equation is
$(r - 1)(r - 2)(r - 3)(r - 4) = 0$

$r^4 - 10r^3 + 35r^2 - 50r + 24 = 0$
Hence, an equation is

$y^{(4)} - 10y^{(3)} + 35y'' - 50y' + 24y = 0.$

Note:
The above answer is not unique. Every nonzero constant multiple ofthe above equation also has the same general solution. However, the indicated equation is the only equation in the standard form that has the given general solution.

Lastly, here is an example of an application of a very simple 4th order nonhomgeneous linear equation that might be familiar to many engineeringstudents.

The static deflection of a uniform beam
Consider a horizontal beam of length L, of uniform cross section and made of homogeneous material, acted upon by a vertical force. The beam is positioned along the x-axis, with its left end at the origin. The deflection of the beam (its vertical displacement relative to the horizontal axis) at any point x is given by, according to the *Euler–Bernoulli beam equation**,

$$EI\,u^{(4)} = W(x), \qquad 0 < x < L.$$

The positive constants E and I are, respectively, the *Young's modulus* of elasticity of the material of the beam, and the beam's cross-sectional moment of inertia about the horizontal axis (i.e., the *second moment of the cross-sectional area*). The value of the product EI is a measurement of the beam's stiffness. The forcing function W describes the load/force that the beam bears per unit length. If the beam is not bearing external load, then $W(x) = w$, which is the weight-density of the beam itself (acting downwards,which is the positive direction per our usual convention). The equation then becomes

$EIu^{(4)} = w.$

It is a (very) simple 4th order nonhomgeneous linear equation. It could be solved simply by integrating both sides four times with respect to x.
However it is certainly more illustrative for our purpose to solve it using the general procedure that we have learned, namely by characteristic equation and undetermined coefficients methods to obtain both parts of the solution ofthis nonhomogeneous linear equation, in the form of $u = u_c + U$.

Its characteristic equation is $EIr^4 = 0$, which has zero as a (quadruple) root.The complementary solution is, therefore, $u_c = C_1 + C_2x + C_3x^2 + C_4x^3$.
The particular solution can be found using the method of Undetermined Coefficients that we have already learned. What form would the particularsolution of the equation take?

The general solution of this deflection equation is

$$u(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + \frac{w}{24EI} x^4 \quad .$$

The graph of the deflection function is called the *deflection curve* or *elasticcurve* of the beam.

Another interesting aspect of this problem is that this equation does not come with initial conditions. Instead, it comes paired with four boundary conditions – describing the physical conditions at the two ends, at $x = 0$ and$x = L$, of the beam. For example, if the beam is securely embedded into walls on both ends, the deflection function above must satisfy the boundaryconditions:

$$u(0) = u'(0) = u(L) = u'(L) = 0.$$

In this case the four conditions tell us that the displacement, $u(x)$, is zero at both ends, and that the slope of the beam, $u'(x)$, is also zero at the two ends –hence the beam is securely fixed at both ends (by embedding into the walls).Notice that there are four conditions, which are necessary to solve the four unknown coefficients present in the general solution of this fourth order equation.

Being a fourth order equation, the boundary conditions in a beam problem frequently involve u'' and u'''. Physically, u'' represents the *bending moment* and u''' represents the *shear force* experienced by the beam at a given point.

For a *simply supported beam*, as another example, where a beam is eitherpinned or hinged at both ends (having no displacement nor resistance to rotation at the two ends), the required boundary conditions are, therefore,

$$u(0) = u''(0) = u(L) = u''(L) = 0.$$

We will study simple boundary value problems, in a quite different context,later in the course.

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