

Some Applications of the Cayley-Hamilton Theorem in Matrices

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Abstract

In this study, by applying Cayley-Hamilton theorem, different order powers, square root, cube root, $\ln(A)$ and $\exp(A)$ operations of an A square matrix are performed. In some studies published on livestock data, the matrix was formed by taking the correlation coefficients showing the relationships between some morphological characteristics of the animals. The polynomials and functions of the generated correlation matrices were determined. The eigenvalues of the matrices are found with the Cayley-Hamilton theorem, and the higher order polynomials and functions of these matrices are obtained. High-order matrix polynomials, which are very difficult to obtain analytically, can be solved by the Cayley-Hamilton theorem.

Keywords: Cayley-Hamilton theorem, matrix, eigenvalues.

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I. Introduction

The Cayley-Hamilton theorem was explored by the prominent mathematicians Arthur Cayley and William Hamilton. The Cayley-Hamilton theorem consents us to compute functions of matrices like the matrix exponent or matrix logarithm, in a simplistic and uniform way, by making use of the fact that a matrix taken to a certain power, can be expressed as a sum of lower powers of the original matrix [1]. Therefore, the powers of square matrices, such as their logarithm, exponential expression, inverse and trigonometric value, can be easily calculated with this theorem. There are publications made by some researchers on this subject [2-5].

[6] proposed Leverrier-Faddeev algorithm and the block Cayley-Hamilton theorem for computation of the outer inverse with prescribed range and null space and the Drazin inverse of block matrix. These algorithms were implemented using symbolic and functional possibilities of the packages Mathematica and using numerical possibilities of Matlab.

In the Cayley-Hamilton theorem has been extended to the fractional standard and discrete-time linear systems [7, 8].

In most cases, the solution to the inverse problem starts by placing the given requested eigenvalues along the main diagonal entries of a diagonal matrix Λ [9]. Frequently, the research on inverse eigenvalue problems has focused on the existence, uniqueness and computability of a solution. Other studies are typically variations on those described, also partially described problems where not all spectral information is known. These types of problems have been considered in [10, 11], and are of interest for problems requiring only certain frequencies to be specifically determined.

The Cayley-Hamilton and Frobenius theorems via the Laplace transform were performed by [12]. In other study, the representation theory of solvable Lie algebras by generalized Cayley-Hamilton theorem was applied [13].

Extending of the classical Cayley-Hamilton theorem to continuous time linear systems with delays was presented in [14] and the extension to nonlinear time varying systems with square and rectangular system matrices was propounded in [14].

In this study, it is aimed to apply the Cayley-Hamilton theorem on square matrices and to compare the results by taking different powers of the correlation matrices.

II. Material and Method

Cayley-Hamilton Theorem: Let A be a square n by n matrix. Its characteristic equation is

$$p(\lambda) = \det(A - \lambda I) = 0$$

Thus, by applying the theorem, matrix A satisfies its own characteristic polynomial, $p(A)=0$ [15].

This is a polynomial equation of degree n .

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \dots + a_1\lambda + a_0 = 0$$

The roots λ_i of the characteristic equation are the eigenvalues of A. Then the matrix satisfies its own characteristic equation.

$$A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \dots + a_1A + a_0I = 0$$

The proof follows from the definition of eigenvalue $Av = \lambda v$ here v is an eigenvector of A. Eigenvectors are characterized by a direction only, and are independent on their length. If v is an eigenvector then also λv is an eigenvector and therefore $A^2v = \lambda^2v$ and so on [1].

If its equation is the characteristic equation of matrix A, given a (2*2) square matrix, $\det(A-\lambda I)=0$ is calculated to find its eigenvalues. There are two eigenvalues. According to this,

$$\begin{aligned} f(A) &= a_0I + a_1A \\ f(\lambda_1) &= a_0I + a_1\lambda_1 \\ f(\lambda_2) &= a_0I + a_1\lambda_2 \end{aligned}$$

Here λ_1 and λ_2 are different eigenvalues.

Let it be a square matrix of size (3*3). To obtain the eigenvalues of this matrix, $\det(A-\lambda I)=0$ is calculated.

$$f(A) = a_0I + a_1A + a_2A^2$$

is,

$$\begin{aligned} f(\lambda_1) &= a_0I + a_1\lambda_1 + a_2\lambda_1^2 \\ f(\lambda_2) &= a_0I + a_1\lambda_2 + a_2\lambda_2^2 \\ f(\lambda_3) &= a_0I + a_1\lambda_3 + a_2\lambda_3^2 \end{aligned}$$

Here λ_1, λ_2 and λ_3 are three different eigenvalues [16].

III. Results and Discussion

For a square matrix of (2*2) given below, \sqrt{A} , $\sqrt[3]{A}$, A^{-5} ve A^{100} can be calculated.

$$A = \begin{bmatrix} 8 & 3 \\ 0 & 1 \end{bmatrix}$$

in the matrix given in the figure, firstly $\det(A-\lambda I)=0$ is calculated for each operation and the characteristic roots (eigenvalues) of this matrix are determined. Here I is the unit matrix. Well,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

form. To find the eigenvalues,

$$\det\left(\begin{bmatrix} 8 & 3 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

is calculated.

$$\begin{aligned} &\begin{vmatrix} 8 & 3 \\ 0 & 1 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ &\begin{vmatrix} 8 & 3 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} \\ &\begin{vmatrix} 8 - \lambda & 3 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \end{aligned}$$

By taking the determinant of this matrix, $(8 - \lambda)(1 - \lambda) - 0 = 0$

This equation contains, $\lambda_1 = 8$ and $\lambda_2 = 1$. These values are respectively,

$$\begin{aligned} f(\lambda_1) &= \alpha_0I + \alpha_1\lambda_1 \\ f(\lambda_2) &= \alpha_0I + \alpha_1\lambda_2 \end{aligned}$$

All matrix calculations can be done by putting them in their places in the equations. To calculate \sqrt{A} ,

$$\begin{aligned} \sqrt{A} &= A^{1/2} \\ f(8) &= \alpha_0 + 8\alpha_1 = 8^{1/2} \\ f(1) &= \alpha_0 + \alpha_1 = 1^{1/2} \\ \alpha_0 + 8\alpha_1 &= \sqrt{8} \\ \alpha_0 + \alpha_1 &= 1 \end{aligned}$$

From the solution of this system of equations,

$$\alpha_1 = \frac{\sqrt{8} - 1}{7}$$

and

$$\alpha_0 = \frac{8 - \sqrt{8}}{7}$$

is found.

$$\sqrt{A} = f(A) = \frac{8 - \sqrt{8}}{7} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\sqrt{8} - 1}{7} \begin{bmatrix} 8 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\sqrt{A} = \begin{bmatrix} \sqrt{8} & \frac{3(\sqrt{8} - 1)}{7} \\ 0 & 1 \end{bmatrix}$$

available as.

Calculation of $\sqrt[3]{A}$

$$f(8) = \alpha_0 + 8\alpha_1 = 8^{1/3}$$

$$f(1) = \alpha_0 + \alpha_1 = 1^{1/3}$$

$$\alpha_0 + 8\alpha_1 = 2$$

$$\alpha_0 + \alpha_1 = 1$$

$$7\alpha_1 = 1$$

$$\alpha_1 = 1/7$$

$$\alpha_0 + 8 * 1/7 = 2$$

$$\alpha_0 = 2 - \frac{8}{7} = \frac{6}{7}$$

$$\alpha_0 = \frac{6}{7} \text{ and } \alpha_1 = \frac{1}{7}$$

By substituting it in the equation, the desired calculation is obtained.

$$f(A) = a_0I + a_1A = \frac{6}{7} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{7} \begin{bmatrix} 8 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\sqrt[3]{A} = A^{1/3} = \begin{bmatrix} 2 & \frac{3}{7} \\ 0 & 1 \end{bmatrix}$$

as obtained.

Calculation of A^{-5}

From this equation, $\lambda_1 = 8$ ve $\lambda_2 = 1$ was found. The eigenvalues of the matrix are, $\lambda_1 = 8$ and $\lambda_2 = 1$. These values are, respectively,

$$f(\lambda_1) = \alpha_0I + \alpha_1\lambda_1$$

$$f(\lambda_2) = \alpha_0I + \alpha_1\lambda_2$$

eşitliklerinde yerlerinde konularak A^{-5} hesaplanabilir.

A^{-5} was calculated by putting them in their places in the equations.

$$f(8) = \alpha_0 + 8\alpha_1 = 8^{-5}$$

$$f(1) = \alpha_0 + \alpha_1 = 1^{-5}$$

$$\alpha_0 + 8\alpha_1 = 8^{-5}$$

$$\alpha_0 + \alpha_1 = 1$$

This system of equations is solved.

$$\alpha_1 = \frac{8^{-5} - 1}{7}$$

$$\alpha_0 = \frac{8 - 8^{-5}}{7}$$

$$f(A) = A^{-5} = \frac{8 - 8^{-5}}{7} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{8^{-5} - 1}{7} \begin{bmatrix} 8 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A^{-5} = \begin{bmatrix} \frac{1}{8^5} & \frac{3}{7} \left(\frac{1}{8^5} - 1 \right) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -0.4286 \\ 0 & 1 \end{bmatrix}$$

is written as.

Calculation of A^{600}

Since it's the same matrix, the eigenvalues are, the eigenvalues, $\lambda_1 = 8$ and $\lambda_2 = 1$.

$$f(8) = \alpha_0 + 8\alpha_1 = 8^{600}$$

$$f(1) = \alpha_0 + \alpha_1 = 1^{600}$$

From the solution of this system of equations,

$$\alpha_0 + 8\alpha_1 = 8^{600}$$

$$\alpha_0 + \alpha_1 = 1$$

$$\alpha_0 = \frac{8 - 8^{600}}{7}$$

and

$$\alpha_1 = \frac{8^{600} - 1}{7}$$

obtained. When these are put in the function,

$$f(A) = A^{600} = \frac{8 - 8^{600}}{7} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{8^{600} - 1}{7} \begin{bmatrix} 8 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A^{600} = \begin{bmatrix} \frac{8 - 8^{600} + 8 * 8^{600} - 8}{7} & \frac{3 * (8^{600} - 1)}{7} \\ 0 & \frac{8 - 8^{600} + 8^{600} - 1}{7} \end{bmatrix}$$

$$A^{600} = \begin{bmatrix} 8^{600} & \frac{3}{7}(8^{600} - 1) \\ 0 & 1 \end{bmatrix}$$

is found.

Calculation of e^A

The eigenvalues are, $\lambda_1 = 8$ and $\lambda_2 = 1$.

$$f(8) = \alpha_0 + 8\alpha_1 = e^8$$

$$f(1) = \alpha_0 + \alpha_1 = e^1$$

$$\alpha_0 + 8\alpha_1 = e^8$$

$$\alpha_0 + \alpha_1 = e$$

From this system of equations,

$$\alpha_1 = (e^8 - e)/7$$

$$\alpha_0 = \frac{8e - e^8}{7}$$

values are obtained. These values are substituted in the function.

$$f(A) = e^A = \frac{8e - e^8}{7} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{e^8 - e}{7} \begin{bmatrix} 8 & 3 \\ 0 & 1 \end{bmatrix}$$

$$e^A = \begin{bmatrix} e^8 & \frac{3 * (e^8 - e)}{7} \\ 0 & e \end{bmatrix}$$

is calculated as.

Example

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 0 & 10 \\ 2 & 5 & 1 \end{bmatrix}$$

is given.

Calculation of A^{10}

$$\det(A - I\lambda) = 0$$

$$\det \left(\begin{bmatrix} 1 & 0 & 4 \\ 3 & 0 & 10 \\ 2 & 5 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

$$\det \left(\begin{bmatrix} 1-\lambda & 0 & 4 \\ 3 & -\lambda & 10 \\ 2 & 5 & 1-\lambda \end{bmatrix} \right) = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 4 \\ 3 & -\lambda & 10 \\ 2 & 5 & 1-\lambda \end{vmatrix} = 0$$

When the determinant of this matrix is taken and set to zero, when the third degree equation obtained to find the unknown parameter λ is solved, 3 λ values are found.

$$(1 - \lambda)[(1 - \lambda)(-\lambda) - 10 * 5] - 0[3(1 - \lambda) - 2 * 10] + 4[3 * 5 - 2(-\lambda)]$$

Eigenvalues are obtained when the third degree equation obtained from here is solved. These values are the eigenvalues of the matrix. These eigenvalues obtained when solved with the MATLAB program,

$$\lambda_1 = 8.6909, \lambda_2 = -0.1766, \lambda_3 = -6.5143$$

available as. When these values are substituted in the function according to the Cayley-Hamilton theorem,

$$\begin{aligned} f(8.6909) &= a_0I + 8.6909 a_1 + 8.6909^2 a_2 = 8.6909^{10} \\ f(-0.1766) &= a_0I + (-0.1766)a_1 + (-0.1766)^2 a_2 = (-0.1766)^{10} \\ f(-6.5143) &= a_0I + -6.5143a_1 + (-6.5143)^2 a_2 = (-6.5143)^{10} \end{aligned}$$

and by making these calculations, the following system of equations is obtained.

The inverse matrix method can be considered for the solution of a system of linear equations with three unknowns in the form of

$$\begin{aligned} a_0 + 8.6909 a_1 + 75.53174 a_2 &= 2458371590 \\ a_0 - 0.1766a_1 + 0.03 a_2 &= 0 \\ a_0 - 6.5143 a_1 + 42.6341 a_2 &= 137618731.52 \\ AX &= B \end{aligned}$$

When a system of linear equations is expressed as a matrix, X multiplies both sides of the equation by A^{-1} to find the unknown variables. Briefly,

$$\begin{aligned} A^{-1}AX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

the equation has unknown roots.

$$\begin{bmatrix} 1 & 8.6909 & 75.53174 \\ 1 & -0.1766 & 0.03 \\ 1 & -6.5143 & 42.6341 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2458371590 \\ 0 \\ 137618731.52 \end{bmatrix}$$

From here, to find the unknowns a_0, a_1 and a_2 ,

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 8.6909 & 75.53174 \\ 1 & -0.1766 & 0.03 \\ 1 & -6.5143 & 42.6341 \end{bmatrix}^{-1} * \begin{bmatrix} 2458371590 \\ 0 \\ 137618731.52 \end{bmatrix}$$

it is calculated. When calculating with MATLAB program,

$$a_0 = 18\ 869\ 641.62$$

$$a_1 = 110\ 182\ 532.3$$

$$a_2 = 19\ 619\ 786.05$$

is obtained.

$$f(A) = a_0I + a_1A + a_2A^2$$

Here I matrix is the unit matrix.

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 0 & 10 \\ 2 & 5 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 9 & 20 & 8 \\ 23 & 50 & 22 \\ 19 & 5 & 59 \end{bmatrix}$$

$$a_0I + a_1A + a_2A^2 = 18869641.62 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 110182532.3 \begin{bmatrix} 1 & 0 & 4 \\ 3 & 0 & 10 \\ 2 & 5 & 1 \end{bmatrix} + 19619786.05 \begin{bmatrix} 9 & 20 & 8 \\ 23 & 50 & 22 \\ 19 & 5 & 59 \end{bmatrix}$$

is calculated.

$$A^{10} = \begin{bmatrix} 305582729 & 393233060 & 596653992 \\ 781738595 & 1001870310 & 1530958286 \\ 593251791 & 647509225 & 1288665379 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 6 & 5 & 1 \\ 4 & 10 & 1 \\ 4 & 3 & 1 \end{bmatrix}$$

is given.

Calculation of \sqrt{A}

It was calculated as $\text{Det}(A)=14$. Eigenvalues

$$\lambda_1 = 13.4287, \lambda_2 = 3.2505, \lambda_3 = 0.3207$$

was calculated as. When these values are put into the function and calculated in accordance with the Cayley-Hamilton theorem,

$$\begin{aligned} f(13.4287) &= a_0I + 13.4287a_1 + 180.33 a_2 = 3.665 \\ f(3.2505) &= a_0I + (3.2505)a_1 + 10.566 a_2 = 1.8 \\ f(0.3207) &= a_0I + 0.3207a_1 + 0.1028a_2 = 0.566 \end{aligned}$$

The inverse matrix method is used to solve a system of linear equations with three unknowns.

$$AX = B$$

as a system of linear equations in the form of

$$X = A^{-1}B$$

The unknown roots of the equation are investigated.

$$\begin{bmatrix} 1 & 13.4287 & 180.33 \\ 1 & 3.2505 & 10.566 \\ 1 & 0.3207 & 0.1028 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3.665 \\ 1.8 \\ 0.566 \end{bmatrix}$$

From here, to find the unknowns a_0, a_1 and a_2 ,

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 13.4287 & 180.33 \\ 1 & 3.2505 & 10.566 \\ 1 & 0.3207 & 0.1028 \end{bmatrix}^{-1} * \begin{bmatrix} 3.665 \\ 1.8 \\ 0.566 \end{bmatrix}$$

It is calculated. When calculating with MATLAB program,

$$a_0=0.412$$

$$a_1=0.486$$

$$a_2=-0.0182$$

is obtained.

$$f(A) = a_0I + a_1A + a_2A^2$$

Here I is the unit matrix.

$$A = \begin{bmatrix} 6 & 5 & 1 \\ 4 & 10 & 1 \\ 4 & 3 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 60 & 83 & 12 \\ 68 & 123 & 15 \\ 40 & 53 & 8 \end{bmatrix}$$

$$a_0I + a_1A + a_2A^2 = 0.412 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0.486 \begin{bmatrix} 6 & 5 & 1 \\ 4 & 10 & 1 \\ 4 & 3 & 1 \end{bmatrix} - 0.0182 \begin{bmatrix} 60 & 83 & 12 \\ 68 & 123 & 15 \\ 40 & 53 & 8 \end{bmatrix}$$

is calculated.

$$\sqrt{A} = \begin{bmatrix} 2.2360 & 0.9194 & 0.2676 \\ 0.7064 & 3.0334 & 0.213 \\ 1.2160 & 0.4934 & 0.7524 \end{bmatrix}$$

Similar to these applications, various matrix calculations can be made by applying the Cayley-Hamilton theorem by using correlation matrices in studies in the field of livestock.

In a study, correlation matrix for body measurements of adult Inyambo cows was examined [17]. The correlation coefficients between the body morphology characteristics of Inyambo cows are presented in Table 1.

Table 1. Correlation matrix

	BW	HG	BL	DW
BW	1	0.98	0.63	0.39
HG	0.98	1	0.63	0.42
BL	0.63	0.63	1	0.42
DW	0.39	0.42	0.42	1

BW: Body weight, HG: Heart girth, BL: Body length, DW: Dewlap width.

The correlation matrix shown in Table 1 is also a symmetric matrix. This matrix is written as a (4*4) square matrix as follows.

$$A = \begin{bmatrix} 1 & 0.98 & 0.63 & 0.39 \\ 0.98 & 1 & 0.63 & 0.42 \\ 0.63 & 0.63 & 1 & 0.42 \\ 0.39 & 0.42 & 0.42 & 1 \end{bmatrix}$$

For this matrix, the matrices A^{25} , A^{-3} , \sqrt{A} , $\ln(A)$ and $\exp(A)$ can be calculated.

Calculation of A^{25}

It was calculated as $\text{Det}(A)=0.0182$. Eigenvalues

$$\lambda_1 = 0.0194, \lambda_2 = 0.4569, \lambda_3 = 0.737, \lambda_4 = 2.7868$$

was calculated as. When these values are put into the function and calculated in accordance with the Cayley-Hamilton theorem,

$$f(0.0194) = a_0 + 0.0194a_1 + 0.000376 a_2 + 0.0000073a_3 = 0$$

$$f(0.4569) = a_0 + 0.4569a_1 + 0.208758a_2 + 0.095381352a_3 = 0.00000000313$$

$$f(0.737) = a_0 + 0.737a_1 + 0.543169 a_2 + 0.400315553a_3 = 0.000486057$$

$$f(2.7868) = a_0 + 2.7868a_1 + 7.66625 a_2 + 21.64299732a_3 = 134166795360.569$$

The inverse matrix method is used to solve a system of linear equations with three unknowns.

$$AX = B$$

as a system of linear equations in the form of

$$X = A^{-1}B$$

The unknown roots of the equation are investigated.

$$\begin{bmatrix} 1 & 0.0194 & 0.000376 & 0.0000073 \\ 1 & 0.4569 & 0.208758 & 0.095381352 \\ 1 & 0.737 & 0.543169 & 0.400315553 \\ 1 & 2.7868 & 7.66625 & 21.64299732 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.00000000313 \\ 0.000486057 \\ 134166795360.569 \end{bmatrix}$$

From here, to find the unknowns a_0, a_1, a_2 and a_3 ,

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0.0194 & 0.000376 & 0.0000073 \\ 1 & 0.4569 & 0.208758 & 0.095381352 \\ 1 & 0.737 & 0.543169 & 0.400315553 \\ 1 & 2.7868 & 7.66625 & 21.64299732 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0.00000000313 \\ 0.000486057 \\ 134166795360.569 \end{bmatrix}$$

It is calculated. When calculating with MATLAB program,

```
a=[1 0.0194 0.000376 0.0000073;1 0.4569 0.208758 0.095381352;1 0.737 0.543169 0.400315553;1 2.7868 7.66625 21.64299732]
```

a =

```
1.0000 0.0194 0.0004 0.0000
1.0000 0.4569 0.2088 0.0954
1.0000 0.7370 0.5432 0.4003
1.0000 2.7868 7.6662 21.6430
```

```
b=[0 0.00000000313 0.000486057 134166795360.569]
```

b =

```
1.0e+11 *
0 0.0000 0.0000 1.3417
```

k=inv(a)*b'

k =

```
1.0e+10 *
-0.0066
0.3620
-1.2205
1.0059
```

$a_0 = -65717570.06$

$a_1 = 3620262869$

$a_2 = -12204682372$

$a_3 = 10059036989$

is obtained.

$$f(A) = a_0I + a_1A + a_2A^2 + a_3A^3$$

Here I is the unit matrix.

$$A = \begin{bmatrix} 1 & 0.98 & 0.63 & 0.39 \\ 0.98 & 1 & 0.63 & 0.42 \\ 0.63 & 0.63 & 1 & 0.42 \\ 0.39 & 0.42 & 0.42 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2.5094 & 2.5207 & 2.0412 & 1.4562 \\ 2.5207 & 2.5337 & 2.0538 & 1.4868 \\ 2.0412 & 2.0538 & 1.9702 & 1.3503 \\ 1.4562 & 1.4868 & 1.3503 & 1.5049 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6.8336 & 6.8775 & 5.8218 & 4.3509 \\ 6.8775 & 6.9223 & 5.8625 & 4.3966 \\ 5.8218 & 5.8625 & 5.1172 & 3.8364 \\ 4.3509 & 4.3966 & 3.8364 & 3.2644 \end{bmatrix}$$

$$a_0I + a_1A + a_2A^2 + a_3A^3 = -65717570.06 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + 3620262869 \begin{bmatrix} 1 & 0.98 & 0.63 & 0.39 \\ 0.98 & 1 & 0.63 & 0.42 \\ 0.63 & 0.63 & 1 & 0.42 \\ 0.39 & 0.42 & 0.42 & 1 \end{bmatrix} -$$

$$12204682372 \begin{bmatrix} 2.5094 & 2.5207 & 2.0412 & 1.4562 \\ 2.5207 & 2.5337 & 2.0538 & 1.4868 \\ 2.0412 & 2.0538 & 1.9702 & 1.3503 \\ 1.4562 & 1.4868 & 1.3503 & 1.5049 \end{bmatrix} + 10059036989 \begin{bmatrix} 6.8336 & 6.8775 & 5.8218 & 4.3509 \\ 6.8775 & 6.9223 & 5.8625 & 4.3966 \\ 5.8218 & 5.8625 & 5.1172 & 3.8364 \\ 4.3509 & 4.3966 & 3.8364 & 3.2644 \end{bmatrix}$$

is calculated.

$$n = -65717570.06 * i + 3620262869 * a - 12204682372 * a^2 + 10059036989 * a^3$$

n =

```
1.0e+10 *
4.1667 4.1964 3.5930 2.7405
4.1964 4.2264 3.6186 2.7600
3.5930 3.6186 3.0983 2.3632
2.7405 2.7600 2.3632 1.8024
```

Manual analytical solution and MATLAB program and direct solution were found to be very close to each other. The MATLAB program solution is as follows.

$$n2 = a^{25}$$

n2 =

```
1.0e+10 *
4.2034 4.2334 3.6246 2.7646
4.2334 4.2635 3.6505 2.7843
3.6246 3.6505 3.1255 2.3839
2.7646 2.7843 2.3839 1.8183
```

$$A^{25} = \begin{bmatrix} 42033751971 & 42333519267 & 36246084381 & 27646046969 \\ 42333519267 & 42635424379 & 36504576430 & 27843207117 \\ 36246084381 & 36504576430 & 31255326287 & 23839436268 \\ 27646046969 & 27843207117 & 23839436268 & 18183099942 \end{bmatrix}$$

found as.

Likewise if A^{-3} , \sqrt{A} , $\ln(A)$ and $\exp(A)$ are calculated directly with the MATLAB program respectively, Calculated of A^{-3}

$$n3 = a^{(-3)}$$

n3 =

```
1.0e+04 *
6.7644 -6.8264 -0.0738 0.2651
-6.8264 6.8895 0.0739 -0.2675
-0.0738 0.0739 0.0016 -0.0031
0.2651 -0.2675 -0.0031 0.0106
```

$$A^{-3} = \begin{bmatrix} 67643.571 & -68263.796 & -738.228 & 2651.137 \\ -68263.796 & 68894.563 & 739.446 & -2675.462 \\ -738.228 & 739.446 & 16.046 & -30.845 \\ 2651.137 & -2675.462 & -30.845 & 106.482 \end{bmatrix}$$

Calculated of \sqrt{A}

$$n4 = a^{(1/2)}$$

n4 =

```
0.7409 0.5986 0.2700 0.1411
0.5986 0.7362 0.2663 0.1696
0.2700 0.2663 0.9075 0.1805
0.1411 0.1696 0.1805 0.9585
```

$$\sqrt{A} = \begin{bmatrix} 0.7409 & 0.5986 & 0.27 & 0.1411 \\ 0.5986 & 0.7362 & 0.2663 & 0.1696 \\ 0.27 & 0.2663 & 0.9075 & 0.1805 \\ 0.1411 & 0.1696 & 0.1805 & 0.9585 \end{bmatrix}$$

Calculated of ln(A)

To calculate ln(a) in MATLAB program log(a) is written.

n5=log(a)

n5 =

```

0 -0.0202 -0.4620 -0.9416
-0.0202 0 -0.4620 -0.8675
-0.4620 -0.4620 0 -0.8675
-0.9416 -0.8675 -0.8675 0
    
```

$$\ln(A) = \begin{bmatrix} 0 & -0.0202 & -0.4620 & -0.9416 \\ -0.0202 & 0 & -0.4620 & -0.8675 \\ -0.4620 & 0.2663 & 0 & -0.8675 \\ -0.9416 & -0.8675 & -0.8675 & 0 \end{bmatrix}$$

Calculated of exp(A)

n6=exp(a)

n6 =

```

2.7183 2.6645 1.8776 1.4770
2.6645 2.7183 1.8776 1.5220
1.8776 1.8776 2.7183 1.5220
1.4770 1.5220 1.5220 2.7183
    
```

$$\exp(A) = \begin{bmatrix} 2.7183 & 2.6645 & 1.8776 & 1.4770 \\ 2.6645 & 2.7183 & 1.8776 & 1.5220 \\ 1.8776 & 1.8776 & 2.7183 & 1.5220 \\ 1.4770 & 1.5220 & 1.5220 & 2.7183 \end{bmatrix}$$

In these studies, the Cayley-Hamilton theorem has been applied on correlation matrices that consist of different powers of some matrices and relationships between different matrix polynomials and morphological characteristics of animals. Applications of this theorem have been seen in other fields as well. In one of the studies made with this theorem, Cayley-Hamilton theorem was generalized to any polynomial matrix of arbitrary degree with coefficients as square matrices of any order [18]. It was presented an algorithm for computing $A(2)$, T , S inverse as well as an algorithm for computing the Drazin inverse of a given block matrix by [6]. These algorithms were based on the Leverrier-Faddeev algorithm and the block Cayley-Hamilton theorem. They proved theorems which provides these algorithms. They also presented and compared computational complexities of these and standard algorithms. Numerical examples which compare algorithms with corresponding algorithms on element-wise given matrices were presented.

IV. Conclusion

Using the Cayley-Hamilton theory, applications were made on different samples in (2*2) and (3*3) dimensional square matrices. It has been displayed that the known extensions of the Cayley-Hamilton theorem are particular cases of the proposed extension. Application of the desired extension has been characterized by instances in (4*4) dimensional correlation matrices from livestock studies. Considerations can be extended to correlation matrices applied to livestock data.

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