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Some Applications of Euler Graph

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Abstract
The article presents some applications of Euler graphs in identifying whether shapes can be drawn with one line or not, along with interesting practical applied problems related to this problem.
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I. Introduction

In the 18th century, starting from solving the classic problem "Seven Bridges of Konigberg", the great Swiss mathematician Leonhard Euler (1707 - 1783) put forward the most basic ideas that gave birth to graph theory.

Since then, many scientists in the world such as Kirchoff, Jordan, Kelley, Hamilton, Percy Headwood, ... have researched and made this subject increasingly rich and constantly developed in both depth and breadth. One of the reasons why graph theory has developed so strongly is due to its extensive applications in many fields of science, technology and life. In particular, with the advent of computers and the explosion of information technology, graph theory has received more and more attention and is considered a fundamental subject of computer science.

In graph theory, it is impossible not to mention a special type of graph, which is the Euler graph. The study of Euler graphs brings many new discoveries and many applications in life as well as in solving math problems. In fact, many difficult problems can be solved simply by Euler graph theory, especially the problem of seven bridges in Konigsberg city is also easily solved by Euler graphs. The interesting and useful applications of Euler graphs help us to enhance our interest in research and increase our ability to appreciate the beauty of graph theory. A particularly important application of Euler graphs is to solve problems related to drawing shapes with one stroke in mathematics as well as in real life. The following will be the content related to these applications.

II. Contents

2.1. Theoretical basis

2.1.1. Definitions of Euler graphs [1], [2]

Given graph G=(V,E).

An Euler cycle is a cycle that passes through every edge and every vertex of the graph, each edge does not pass more than once.

An Euler path is a path that passes through every edge and every vertex of the graph, each edge does not pass more than once.

A connected graph that contains an Euler cycle is called an Euler graph. A connected graph that contains an Euler path is called a semi-Eulerian graph.

2.1.2. Necessary and sufficient conditions for a graph to have an Euler cycle and path [1], [2], [3]

Theorem 1. (Euler's Theorem): A graph G has an Euler cycle if and only if G is connected and every vertex has an even degree other than 0.

Theorem 2. A graph G has an Euler path if and only if G is connected and has exactly 2 vertices of odd degree.

Theorem 3. Given a graph G with k vertices of odd degree. Then the minimum number of paths covering G is k/2.

2.1.3. Fleury's algorithm for finding an Euler cycle [1], [2]

Input. Graph $G \neq \emptyset$, no isolated vertices.

Output. Euler cycle C of G, or conclude that G does not have an Euler cycle.

Method.

(1) Choose any starting vertex v0. Let v1 := v0, C := (v0). H := G.

(2) If $H = \emptyset$, then conclude that C is an Euler cycle, end.

Otherwise go to step (3).

(3) Choose the next edge:

- In case v1 is a suspended vertex: There exists a unique vertex v2 adjacent to v1. Choose edge (v1, v2). Go to step (4).

- In case v1 is not a suspended vertex:

If all edges belonging to v1 are bridges, then there is no Euler cycle, end.

- Otherwise, choose any edge (v1, v2) that is not a bridge in H. Add vertex v2 to path C. Go to step (4).

(4) Delete the edge just passed, and delete the isolated vertex:

Remove from H the edge (v1, v2). If H has isolated vertices, then remove them from H. Set v1 := v2. Go to step (2).

2.2. Some applications of Euler graphs [1], [2], [4]

2.2.1. The problem of drawing a shape with one stroke

Given a graph G, is it possible to start from a point and draw G so that it passes through each edge exactly once without leaving the graph?

2.2.2. Solving the problem of drawing a shape with one stroke

The graph G can be drawn with one stroke when G is an Euler or semi-Eulerian graph. In the case where G is semi-Eulerian, we must start from one of the two odd-degree vertices and will end at the remaining odd-degree vertex. In the case where the graph is relatively complex, we should use an algorithm to find an Euler cycle. If the graph G has k odd-degree vertices, then G can be drawn with at least k/2 strokes.

For example: Can the following shapes be drawn with one stroke? If not, how many strokes can be drawn at least?

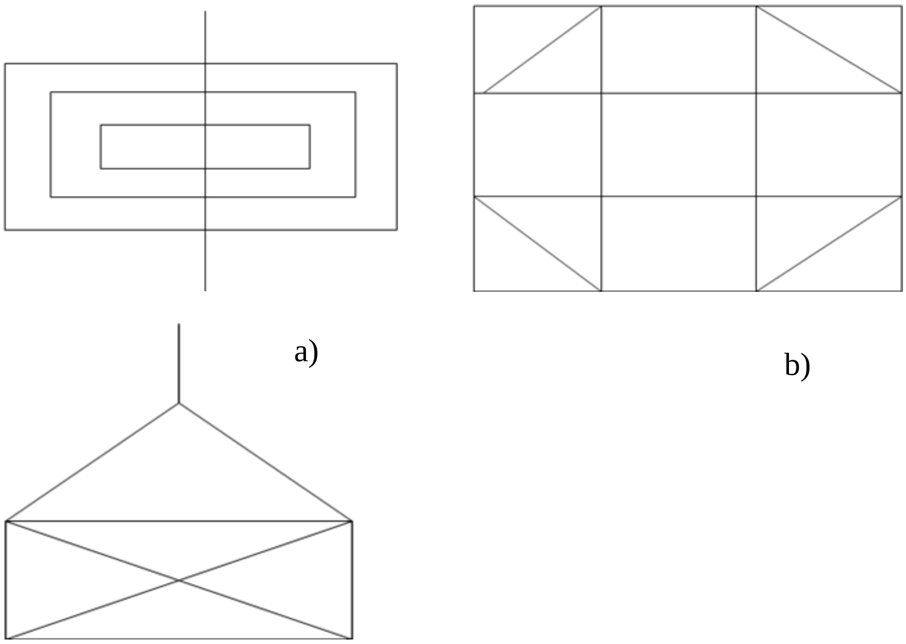

Figure 1

Solution: The graph in Figure 1a) has 2 vertices of odd degree, so according to Theorem 2 it can be drawn with 1 line. Note that when drawing, it must start from 1 of the 2 vertices of odd degree. The graph in Figure 1b) can be drawn with 1 line because all vertices have even degree, not 0. The graph in Figure 1c) cannot be drawn with 1 line because it has 4 vertices of odd degree, it must be drawn with at least 2 lines.**2.2.3. 2.2.3. Some applied problems [1], [3], [4], [5], [6]**

Problem 1: The Seven Bridges of Konigsberg

The city of Konigsberg, Germany (now Kaliningrad, Russia) is located on the Pregel River, consisting of two large islands connected to each other and to the mainland by seven bridges. The question is whether it is possible to follow a route that crosses each bridge exactly once and then returns to the starting point.

In 1736, Euler proved that the problem has no path in the language of graphs. He represented the 2 islands and 2 river banks by 4 points and the 7 bridges by edges connecting the points, resulting in a graph as shown in Figure 4.

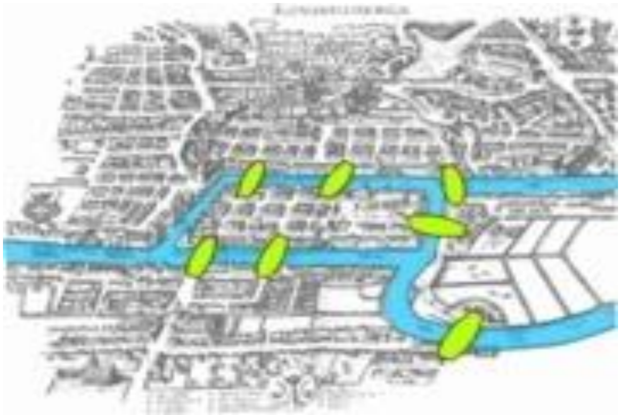

Figure 2

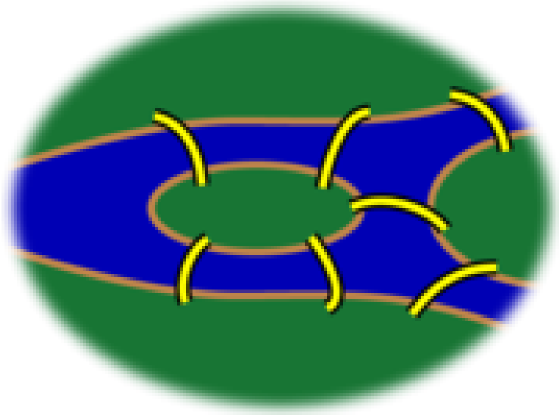

Figure 3

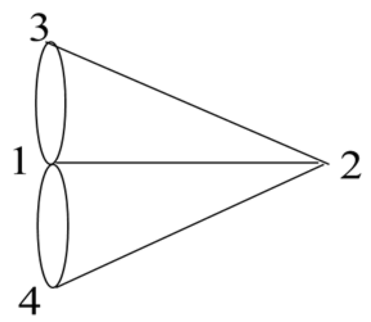

Figure 4

Crossing 7 bridges is equivalent to drawing the graph of Figure 4 with one line. The graph of Figure 4 has 4 vertices of odd degree, so it cannot be drawn with one line. Therefore, we know that this problem does not have a path.

Problem 2. Due to military strategy requirements, an engineering team was ordered to destroy 15 bridges with the condition that after the explosive truck passed over each bridge, it must immediately destroy that bridge and not leave any bridge behind (Figure 5). Is such a route possible?

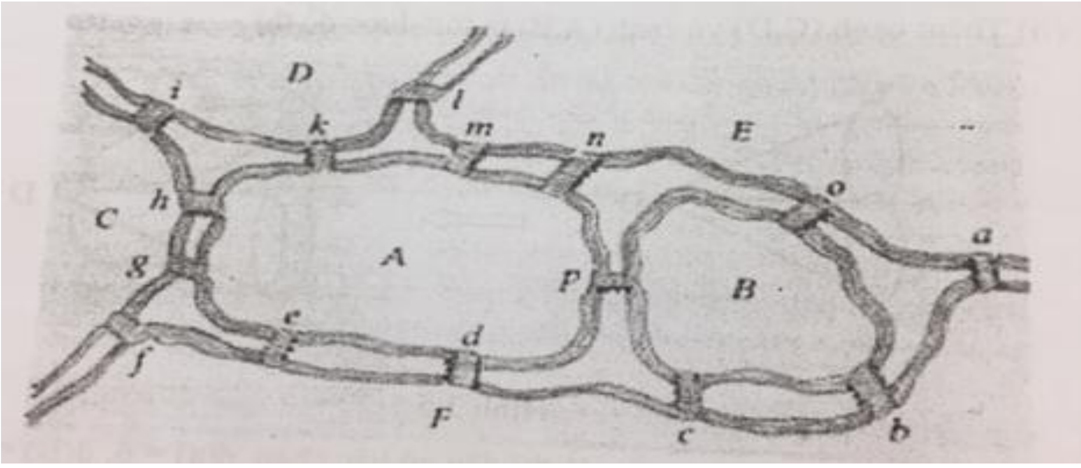

Figure 5

Solution: Construct a graph G (Figure 6) with vertices A, B, C, D, E, F; 15 edges a, b, c, d, e, f, g, h, i, k, l, m, n, o, p. The problem becomes whether there exists an Euler path in G?

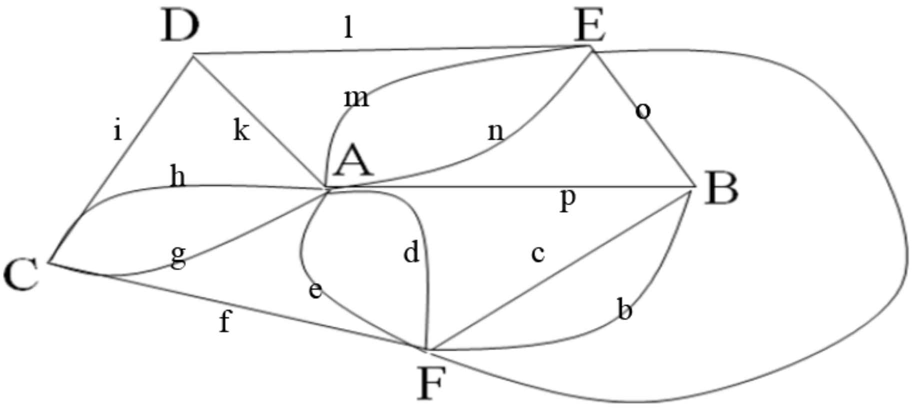

Figure 6

The graph above (Figure 6) has 2 odd-degree vertices D and E, so it can be drawn with 1 stroke. Starting from one of the two odd-degree vertices D or E, we can use the Fleury algorithm to find an Euler path that satisfies the requirements of the problem. For example, a route is: D, i, C, h, A, g, C, f, e, A, d, F, c, B, b, F, a, E, n, A, p, B, o, E, m, A, k, D, l, E.

Problem 3. A king built a castle to hide his treasure. People found a diagram of the castle (Figure 7) with instructions: to find the treasure, just go from one of the outermost rooms (numbers 1, 2, 6, 10, ...), go through all the doors to get the password, each door only once; the treasure is hidden in the last room. Find out where the treasure is hidden?

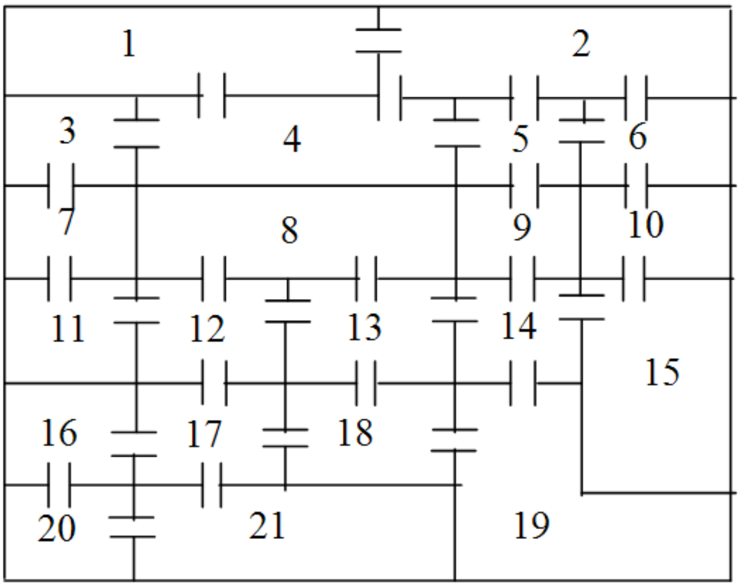

Figure 7

Consider the graph G (Figure 8) in which each vertex corresponds to a room of the castle and two vertices are adjacent if the two corresponding rooms have a direct connecting door. According to the requirements of the problem, G is a simple graph with the following 21 vertices:

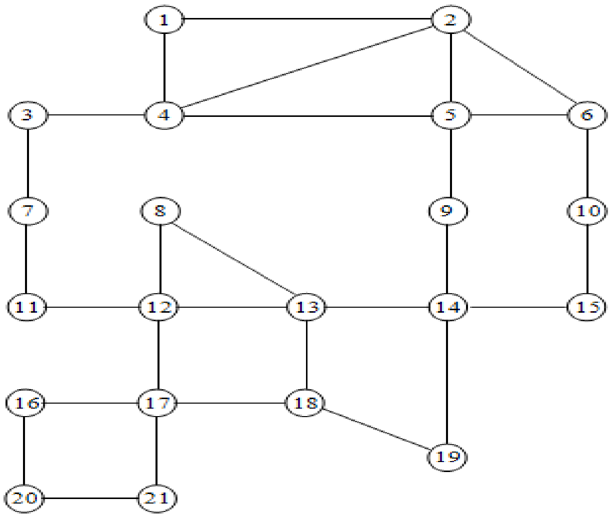

Figure 8

The graph above (Figure 8) has exactly two vertices of odd degree, so it can be drawn with one stroke. Thus, starting from room 6, using Fleury's algorithm, passing through all the doors of the rooms, we will find the treasure in room 18.

Problem 4. Given $n > 2$ points on a circle, each pair of points is connected by a line segment. Is it possible to quickly draw all of these lines so that the end point of the first line segment coincides with the beginning point of the second line segment, the end point of the second line segment coincides with the beginning point of the third line segment, …, the end point of the last line segment coincides with the beginning point of the first line segment?

(Polish Excellent Student Competition, 1961 – 1965)

Solution: Represent the graph G with n vertices as n points on a circle, the edges are straight lines connecting those vertices. Assume that all vertices of G have degree $n – 1$ ($n > 2$).

(i) If n is even, then $n – 1$ is odd, so there is no Euler cycle in G, so we cannot draw a curve that satisfies the problem requirements.

(ii) If n is odd, then $n – 1$ is even, so there is an Euler cycle in G. An Euler cycle in the graph is a cycle that satisfies the problem requirements.

Problem 5. Given a figure consisting of 16 line segments (Figure 9). Prove that it is impossible to draw a curve that intersects each line segment exactly once (the endpoints of the curve cannot lie on the line segments and the curve cannot pass through the vertices of the lines).

Figure 9

Solution: The figure above divides the plane into 6 regions: A, B, C, D, E, F (Figure 10).

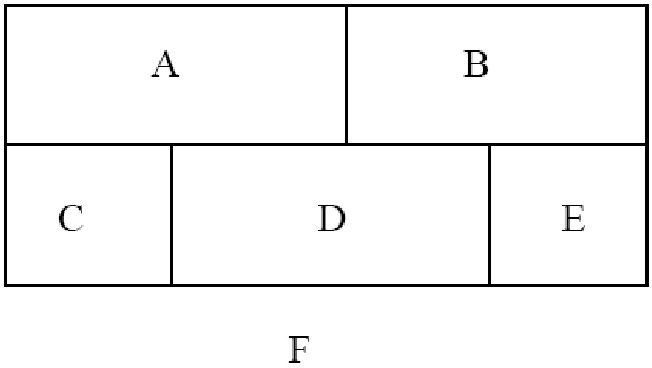

Figure 10

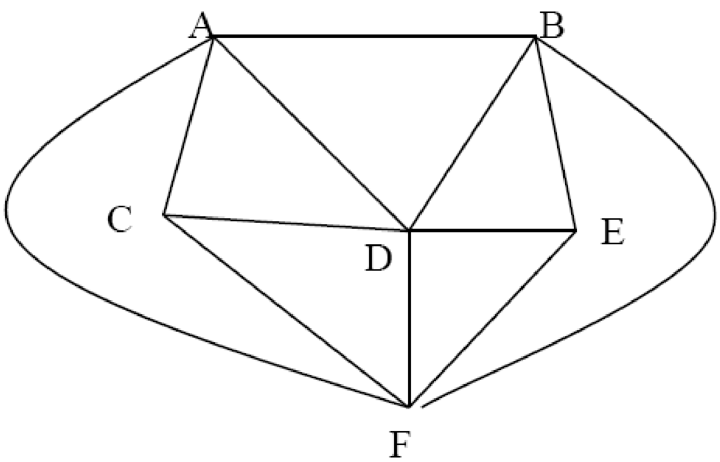

Figure 11

We represent these 6 regions by 6 vertices, two vertices are connected by an edge if the two corresponding regions have a common boundary, we get graph G (Figure 11).

Graph G has 4 vertices of odd degree so there is no Euler cycle in G. So it is impossible to draw a curve that intersects each line segment exactly once.

Problem 6. To open the secret door in an ancient castle, archaeologists must solve the following problem: There are a number of magnets in front of the door, each with a word written on it. The magnets need to be arranged in a row so that the first letter of the word on each magnet (from the second magnet onwards) must match the last letter of the word on the previous magnet. The words on the magnets are: "ear", "rat", "teen", "no", "odd", "unable", "doe", "thru", "ugly", "dot", "need", "day", "yen". Help the archaeologists check whether the magnets can be arranged in a row to open the secret door.

Solution: The set of first and last letters of the magnets is V = {e, r, t, o, d, u, y, n}. Construct a directed polygraph G with the vertex set V. Each word on a magnet corresponds to an arc connecting the letters from the beginning of the word to the last letter of the word (Figure 12). The problem becomes whether there exists an Euler directed path in the directed graph G.

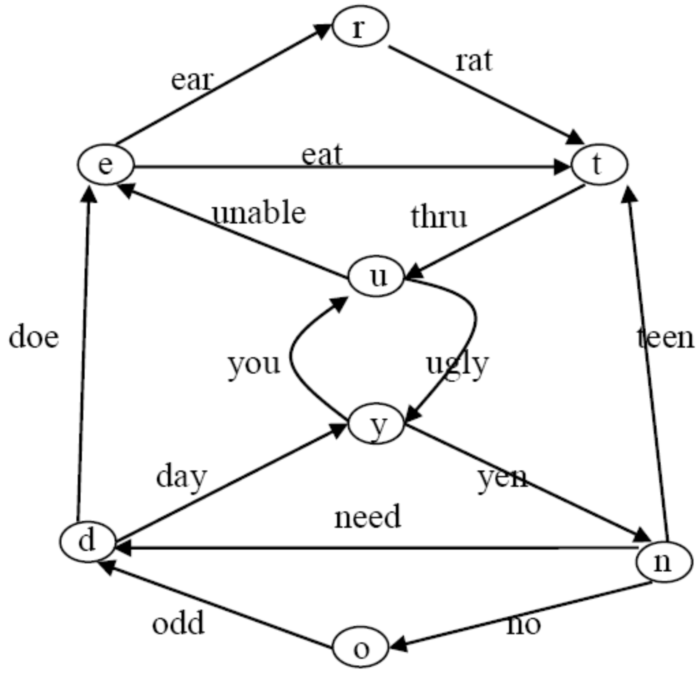

Figure 12

Obviously, the graph is weakly connected and each vertex has half the exit degree and half the entry degree, so the graph has an Euler directed cycle, so it has an Euler directed path. So the archaeologists can open the door to the secret by arranging the magnet pieces in a row. Each Euler path corresponds to an arrangement: ear – rat – teen – no – odd – doe – eat – thru – ugly – yen – need – day – you – unable.

III. Conclusion

The article has introduced an application of Euler graphs in problems of drawing shapes with one line. Through that, we know whether a graph can be drawn with one line, or at least how many lines must be drawn. At the same time, some practical problems related to this issue can be solved easily. Due to the limited scope of the article, I cannot include many interesting practical problems, I hope that the article contributes to bringing mathematics closer to reality.

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