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Chain Necklace Diagrams for Some Iterative Operations on Integers, and Their Properties

Tejmal Rathore
Independent Researcher, Retired from IIT Bombay
Borivali, Mumbai 400066, Bharat
Correspomding author: T S Rathore, tsrathore43@gmail.com

Abstract: *Some iterative operations on integer numbers are reviewed. Their properties are summarized. The operations are shown pictorially by chain-necklace diagrams. The proof of Collatz's conjecture is given.*
Keywords: *Rathore Operation, Kaprekar Operation, Collatz Conjecture, Nollatz Conjecture, Terrance Tao Conjecture*

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I. INTRODUCTION

A) Rathore Iterative Operation (RIO) 1

Let an n-digit number be represented as

$$A_n = d_1 d_2 d_3 \dots d_n \tag{1}$$

and

$$A_{nR} = d_n d_{n-1} \dots d_1 \tag{2}$$

where A_{nR} is the same as A_n but digits are in reverse order. Then the RIO is defined as [1]

$$\mathscr{R}(A_n) = |A_n - A_{nR}|. \tag{3}$$

B) Kaprekar Iterative Operation (KIO)

Kaprekar operation is defined as [2]

$$\mathscr{K}(A_n) = A_n - A_{nR}$$

where A_n is the n-digit number whose digits are arranged such that, it gives the maximum value, and A_{nR} is obtained by reversing the order of the n digits. Obviously, A_n and A_{nR} are, respectively, the maximum and minimum values of the n digit number. Thus, it is a special case of RIO 1.

C) Rathore's Iterative Operation (RIO) 2

The Collatz conjecture (CC) [4] is one of the most famous unsolved problems in mathematics. When Collatz operation is carried out iteratively, it always terminates into 1, no matter which positive integer is chosen to start. The exact origin of CC is obscure. It has circulated by word of mouth in the mathematical community for many years. The problem is traditionally credited to Lothar Collatz. CC is simple to state and apparently intractably hard to solve. Erdős [4] said, "Mathematics may not be ready for such problems.". He also offered US$500 for its solution [4]. Lagarias [4] said, "CC is an extraordinarily difficult problem, completely out of reach of present-day mathematics". People have tried to find out a counter example, but could not succeed [6][7]. However, a logical proof has now appeared in [3].

The Chain-Necklace Diagrams (CNDs) for RIO 1, KIO and RIO 2 and their properties are reviewed in section II. Section III gives the conclusion.

II. CNDs AND THEIR PROPERTIES

A) RIO 1

Let $A_7 = 3390156$, then

$\mathscr{R}(3390156) = |3390156 - 6510933| = 3120777$. Continuing the same procedure on the new number obtained repeatedly, we get the result as shown in the form of a CND in Figure 1.

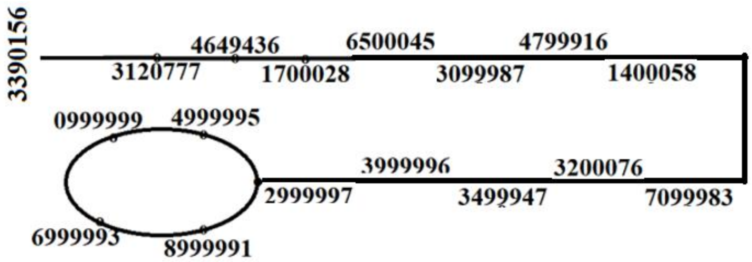


Figure 1: CND for 339015

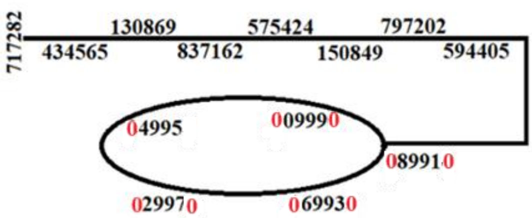


Figure 2: CND for 717282.

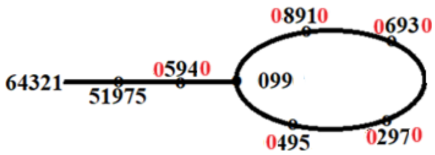
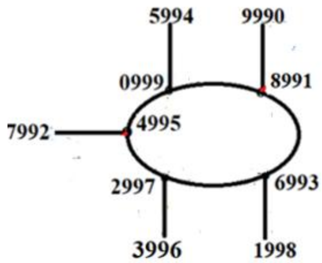
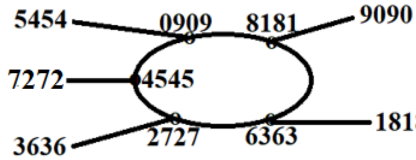


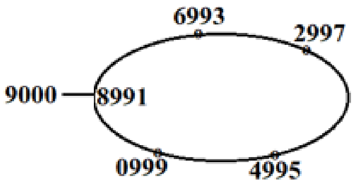
Figure 3: CND for 64321



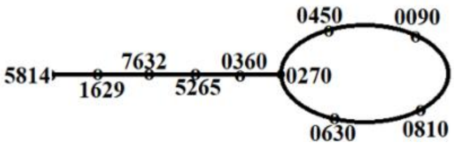
(a) CNDs for 7992, 5994, 9990, 1998, 3996,



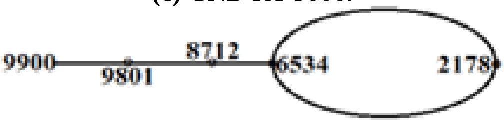
(b) CNDs for 7272, 5454, 9090, 1818, 3636.



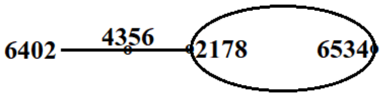
(c) CND for 9000.



(d) CND for 5814.



(e) CND for 9900



(f) CND for 6402

Figure 4: CNDs for 4-digit numbers

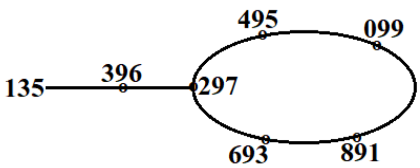


Figure 5: CND for 135.

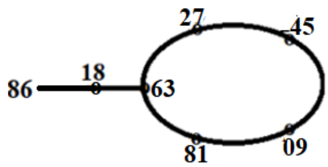


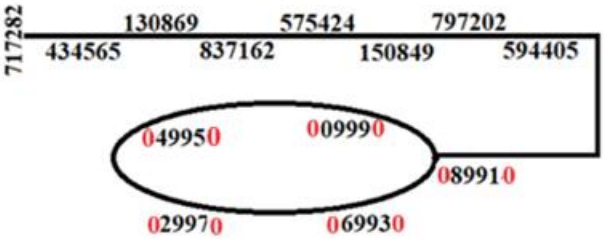
Figure 6: CND for 86.

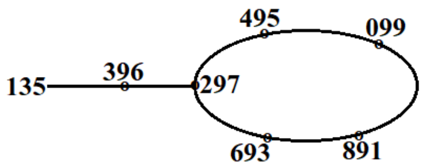
Straight line is the *chain* and the closed loop is the *necklace.* The process goes from left to right on the chain followed by clockwise rotation on the necklace. The given number A_n will be referred as the initial input.

Similarly, CNDs for A_n with different n are shown in Figures 2-6. From these CNDs, we note the following properties.

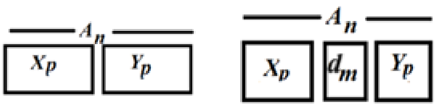
1. A chain is followed by a necklace.
2. All the beads on the chain and the necklace are distinct.

3. For the same n, but different inputs, there may be different CNDs. See Figures 4(a), (b), (c), (d). Though the necklaces are different, each has 5 beads, and their last digits occur in the same cyclic order, i.e., 1,3,7,5,9.
4. If $d_n = d_1$, then subsequent $\mathscr{R}(A_n)$ will have both the first and the last digits 0. See Figures 2 and 3, the digits are shown in red color. Both the digits maybe ignored in further processing, if reduced number of digits are required.
5. If S_l is such that $S_l = S_{lR}$, is centrally located in A_n and the decimal value of the beads to the right of S_l is less (more) than that to the left of S_l in reverse order, then $\mathscr{R}(A_n)$ will have a string R_l with all l beads 9 (0) and centrally located.
   Examples: $\mathscr{R}$(83932323927) = 1099999989, $\mathscr{R}$(81932323936) = 18000000018.
6. Let A_n be partitioned as shown in Figure 9 where d_m is the middle digit, and
$$p = \begin{cases} \dfrac{n}{2}, n \text{ even} \\ \dfrac{(n-1)}{2}, n \text{ odd.} \end{cases} \tag{4}$$
   In A_n (n even (odd), l number of xs (where x is any number from 0 to 9) are inserted after the right of X_p and before the left of Y_p (on either side of the middle digit), the new CND will have $2l$ number of 9s centrally located. The CNDs for 86 and 8xx6 are shown in Figures 6 and 7, respectively, and for 135 and 1xx3xx5 are shown in Figures 5 and 8, respectively.





7. A necklace may have either 5 or 2 beads. These necklaces will be designated as N5 and N2, respectively.
8. In N5-type, All the numbers on the necklace rotate in cyclic order with least significant digits as 1,3,7,5,9 for different inputs.
   In N2 type, the last digits of all the beads on the necklace are even, therefore, the input must have (i) both d_1 and d_n either odd or even and (ii) should be reverse of one of the beads on the necklace. If these conditions do not hold, then input must be followed by a number of iterations until these conditions are met. In Figure 4(c), input 9900 is connected to the necklace after three iterations to give 8712 which is also present in the necklace with digits reversed.
9. The sum of the first and last digits of all the numbers in N5 is 9, and in N2 10.
10. Classification of inputs to the necklace
    (i) Inputs which yield N2 and N5.
    Numbers input to the necklace which give, after 2 or 5 iterations, digits in reversed order as shown in all Figures except Figure 4(c).
    (ii) Inputs which do not follow the property (ii) as shown in Figures 4(c).
    (iii) Inputs which have no number on the chain. If any bead on the necklace becomes the input, then there will be no chain. Examples: 65, 495, 2178.
    (iv) Inputs which have $X_{pR} = Y_p$ do not give any output. These inputs are trivial ones. Examples: 181, 2222, 321123.



(a) (b)

Figure 7: Portioning of A_n (a) n even (b) n odd.

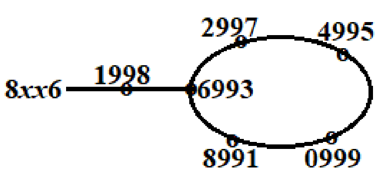


Figure 8: CND for 8xx6

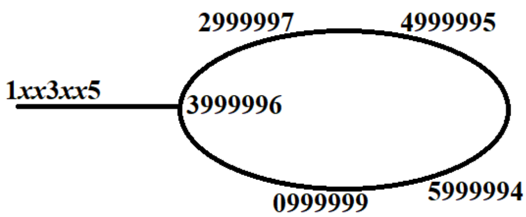


Figure 9: CND for 1xx3xx5

11. For n odd, 9 is centrally located in all the numbers. See CNDs of Figures 2, 4, 6. For n even and the digits repeat, i.e., Y_p is same as X_p, all the beads will also have repetition of digits. See Figure 4(b).
12. For $n > 2$, one possible necklace can be generated by inserting in the middle (n-2) number of xs. 63-693-6993-69993-… However, there are other possible necklaces also. Examples: 2, 3.

Proof of the Properties

Only properties 4 and 7 require proofs, rest are all obvious or trivial.

Proof for property (4)

In the operation $\mathscr{R}(N_n)$

$$d_1 \leq d_n. \tag{5}$$

$$\mathscr{R}(N_n) = (d_1 d_2 d_3 \ldots . d_n) - (d_n d_{n-1} \ldots . d_1)$$
$$= (d_1 10^{n-1} + d_2 10^{n-2} + d_3 10^{n-3} + \ldots . + \; + d_{n-2} 10^2 + d_{n-1} 10^1 \;\; d_n 10^0)$$
$$-(d_n 10^{n-1} + d_{n-1} 10^{n-2} \; + d_{n-2} 10^{n-3} + \ldots . +$$
$$d_3 10^2 + d_2 10^1 + \; d_1 10^0)$$
$$= \sum_i^n (d_i - d_{n+1-i}) \, (10^{n-i} - 10^{i-1}),$$
$$= \sum_i^p (d_i - d_{n+1-i}) \, (10^{n-i} - 10^{i-1}),$$
$$= \sum_i^p (d_i - d_{n+1-i}) \, 10^{i-1} (10^{n-2i+1} - 1) \tag{6}$$

All terms in (6) will be present for n even, and the middle term may be 0 (for $Y_p > X_p$) or 9 (for $Y_p < X_p$). There is a factor $(10^{n-2i+1} - 1)$ which is a multiple of 9. Therefore, the entire expression is divisible by 9.

Proof of Property (7)

Following the rules of subtraction

$$b_1 \; = \begin{cases} |d_1 - d_n| - 1, \text{if there is a borrow by } d_2 \text{ from } d_1 \\ \quad |d_1 - d_n|, \quad\quad \text{if there is no such borrow} \end{cases} \tag{7}$$

$$b_n \; = 10 - |d_1 - d_n|, \tag{8}$$

$$b_1 + b_n \; = \begin{cases} 9, \; \text{if there is a borrow} \\ 10, \quad\quad \text{no borrow} \end{cases} \tag{9}$$

There are three cases:

Case 1: In this case, condition is

$$b_1 + b_n \; = 9 \tag{10}$$

and is valid after each subsequent iteration. Since the sum of all the digits b_1 to b_n should be 9 (as per the property (4) after any iteration, the sum of the digits from b_2 to b_{n-1} should also be 9. Therefore, the admissible combinations $\{b_1,b_n\}$ are

$$\{9,0\},\{8,1\},\{7,2\},\{6,3\},\{5,4\}. \tag{11}$$

Note that all the numbers have the sum of all the digits as 9; so also sum of the first and the last digits. There are in all 5 numbers on the necklace. Thus, there would be 5 groups of numbers whose number just before the necklace will be one of the numbers on the necklace. Same thing happens while starting with other $\{b_1,b_n\}$ combinations of Equation (11). Thus, there are 5 and only 5 numbers on the necklace. The last digits repeat in cyclic order 1,3,7,5,9 for different numbers.

Case 2*:* In this case

$$b_1 + b_n \; = 10 \tag{12}$$

and is valid after each subsequent iteration. It can easily be verified that this case does not exist for n = 2 and 3. For $n > 3$, the possible combinations of $\{(b_1,b_n)\}$ are

$$\{9,1\},\{8,2\},\{7,3\},\{(6,4\},\{5,5\}. \tag{13}$$

The last combination, will give on all the following iterations $b_1 = b_n = 0$. In this case, the neighboring digits b_2,b_{n-1} should be considered. (See Figures 2 and 3).

Since $b_1 + b_n \; = 10$, the sum of all the remaining digits should be 8 after any iteration. The permissible combinations are

$$\{8,0\},\{7,1\},\{6,2\},\{5,3\},\{4,4\}. \tag{14}$$

Let us take 9801. The CND is shown in Figure 4(e). Note that all the numbers have $b_1 + b_4 = 10$. There are in all 2 numbers on the necklace. It can be verified that same thing happens with other b_1,b_4 combinations from Equation (13) and b_2,b_3 from Equation (14). Thus, there are 2 and only 2 numbers 6534 and 2178 on the necklace and both are even and repeat in cyclic order for successive iterations. Thus, there would be 3 groups of numbers which will follow this pattern.

Case 3: In this case the successive iterations may turn out $b_1 + b_n = 9$ or 10. In such a case, the CND will contain N5. The occurrences of 9 or 10 is random depending upon the number A_n. Therefore, a general proof may be difficult. See Examples 1, 2 and 3. The first one has

10,10,9,10,9,9,9,9,10,9,10,1,9,9,10,10,9,10,9,10,9,9,9,9,9,9.

B) KIO

This is illustrated with the following example.

Let the number be 7992. Then A_n = 9972, $\mathscr{K}(9972 - 2799)$ = 7173, $\mathscr{K}$(7731 – 1377) = 6354. On successive operations, we get the CND as shown in Figure 10.

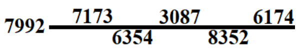
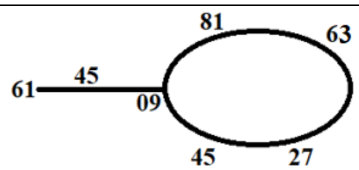
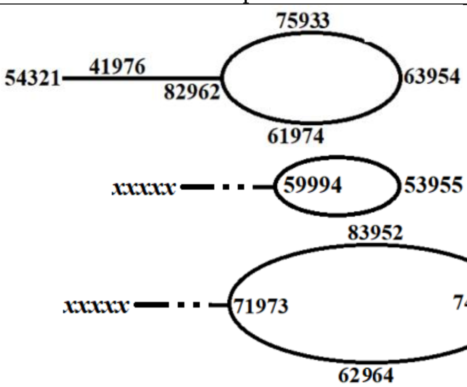
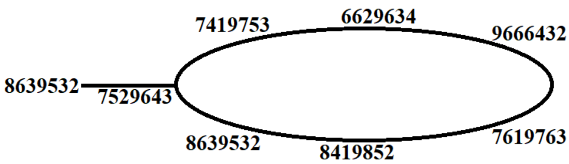


Figure 10: CND for 7992.

Table 1. CNDs for different *n*.

n	**CND**
2	 Chain-necklace
3	Chain which will end up into 495
4	Chain which will end up into 6174
5	 CNDs
6	Chains which will end up into 549945 or 631764
7	
8	Chains which end up into 63317664 or 97508421*
9	Chain which will end up into 987654321*
10	Chain which will end up into 9876543210*

*Reported by Martin Gardner [2]

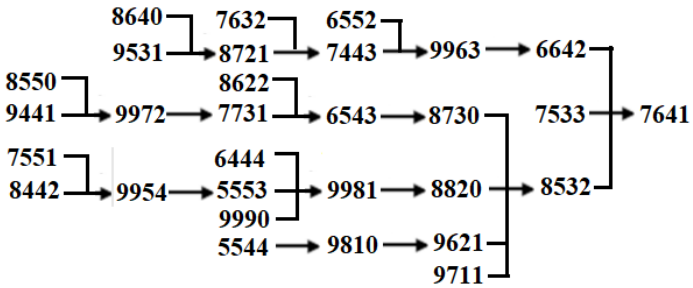


Figure 11: Numbers which will lead to 7641.

Table 1 gives the CNDs for different values of *n*. Note that there can be one chain (*n* = 3,4), multiple chains (*n* = 6,8) and chain-necklace (*n* = 2,5,7).
For all 2-digit numbers, the necklace remains the same. It has been tried for several 5-digit numbers and found that they end up in three different necklaces shown in Table 1.

Possible numbers, number of iterations, and the end integer
Let the largest number formed by rearranging the 4 digits be represented as *abcd*, where

$$9 \le a \ge b \ge c \ge d \ge 0. \tag{15}$$

This leads to

$$(a - d) \ge (b - c). \tag{16}$$

Now

$$abcd - dcba = 9[111(a\text{-}d)\text{-}10(b\text{-}c)]. \tag{17}$$

The 4-digit number must satisfy the following conditions.

1. Numbers should not have the symmetry along the vertical axis.
2. From (17), 9 is a common factor. Therefore, the 4-digit number should be divisible by 9. (This property is true for any *n*).
3. From (15), *a-d* can have values 1-9, and *b-c* can have 0 to 9.
4. Condition $(a - d) \ge (\boldsymbol{b} - \boldsymbol{c})$.
   Discarding the numbers which do not satisfy the above conditions, we find only 30 numbers shown in Figure 11. The figure also shows how they reach 7641. Thus, it is not true that all 4-digit integers lead to 7641, as mentioned in [2]. It is noticed that the maximum number of iterations required is 7 for the numbers 8550, 9441, 7551, 8442 and minimum 1 for 6642,7533,8532.
5. Sum of all the digits in all the numbers 18 except the numbers whose two most significant digits are 99. They are 9990, 9981,9963 whose sum of the digits is 27.

C) *RIO* 2

Start with any *odd* number a_0. Multiply a_0 by an *odd* number *b*. It gives a_0b (odd number). Add an *odd* number *c*. It gives $a_0b + c = d_0$ an *even* number. Divide d_0 continuously by 2 till it ends up in an *odd* number a_1. Repeat the above steps for a_1, in place of a_0. Continue the process. The process goes up and down and eventually ends up with a cycle. It is illustrated for the ith iteration in Figure 11. If the starting number is even, skip the first 3 steps. For convenience, we will designate this entire operation by $\mathscr{R}\{a_0,b,c\} = a_o, d_0, a_1, d_1 \ ... \ d_{i-1}, \overbrace{a_i, d_i, a_{i+1}, ..., a_i}$, where $a_o, d_0, a_1, d_1 \ ... \ d_{i-1}$ represents a chain and $\overbrace{a_i, d_i, a_{i+1}, ..., a_i}$ represents a necklace. For example, for $\mathscr{R}\{9,3,1\}$ the process is shown in Figure 12 and expressed as $\mathscr{R}\{9,3,1\}$ = 9,28,7,22,11,34,17,52,13,40,5,16,$\overbrace{1,4,1}$. Its CND is shown in the second row of Table 2. Even (odd) numbers on the chain are shown above (below) the line and even (odd) numbers outside (inside) the necklace. For various $\mathscr{R}\{a_0,b,c\}$, the results are shown in Table 2.
Figure 12 shows the CND for *i*th iteration. The following recursive relation holds.

$$a_i = \frac{3a_{i-1} + 1}{2^i}. \tag{18}$$

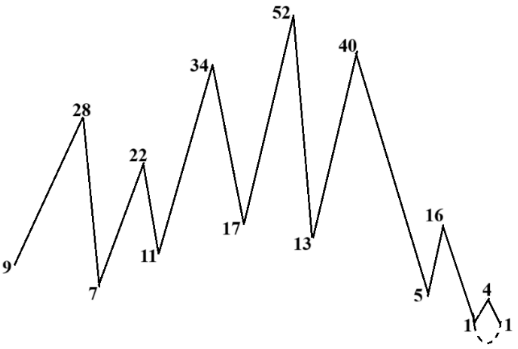


Figure 12: Process of $\mathscr{R}$\{9,3,1\} operation.

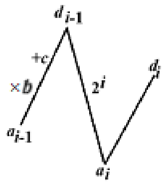
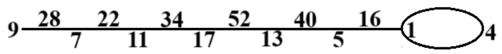
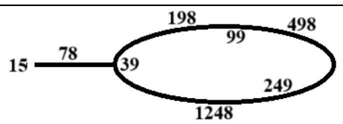
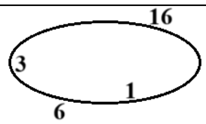
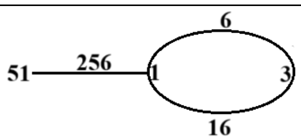
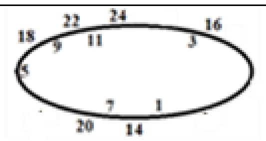
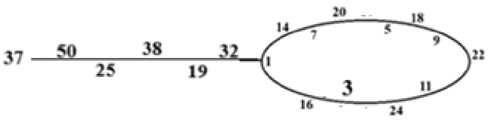
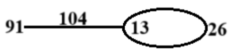


Figure 13: CND for *i*th iteration

Table 2. CNDs for different $\mathscr{R}\{a_0,b,c\}$

$\mathscr{R}\{a_0,b,c\}$	CND	Remarks
$\mathscr{R}\{9,3,1\}$		$\left\{9,28,7,22,11,34,17,26,13,40,5,16,\overbrace{1,4,1}\right\}$
$\mathscr{R}\{181,3,1\}$		$\left\{181,544,17,26,13,40,5,16\ \overbrace{1,4,1}\right\}$
$\mathscr{R}\{15,5,3\}$, a_1 and c have a common factor 3		$\left\{15,\overbrace{39,198,99,498,249,1248,39}\right\}$
$\mathscr{R}\{3,5,1\}$		$\left\{\overbrace{3,16,1,6,3}\right\}$
$\mathscr{R}\{51,5,1\}$		$\left\{51,256,\overbrace{1,6,3,16,1}\right\}$
$\mathscr{R}\{5,1,13\}$ $a_0 < c$		$\left\{\overbrace{5,18,9,22,11,24,3,16,1,14,7,20,5}\right\}$
$\mathscr{R}\{37,1,13\}$ $a_0 > c$		$\left\{\begin{matrix}37,50,25,38,19,32,\\ \overbrace{1,14,7,20,5,18,9,22,11,24,3,16,1}\end{matrix}\right\}$ Numbers on the necklace remain the same in the same cyclic order. All odd numbers < 13 appear in the necklace, so if the input is anyone of them, then there will be no chain. If the input a_0 > 13, all odd numbers > 13 will appear in the chain.
$\mathscr{R}$ $\{m13,1,13\}$ $a_0 = mc$	 $m = 7$	If a_0 is an odd multiples of 13, all numbers are multiples of 13, and the necklace is $\left\{\overbrace{13,26,13}\right\}$

Properties

Some of the obvious properties from Table 2 are the following.

1. The CND always terminates in a necklace. Chain may or may not exist, Examples $\mathscr{R}\{7,5,1\}$, $\mathscr{R}\{5,1,13\}$.
2. Consider the CND of the ith iteration shown in Fig. 11. The recursive relation is

$$a_i = \frac{ba_{i-1} + c}{2^i}, i \geq 1. \tag{19}$$

   where 2^i has a value which results a_i as an odd number.
3. Whenever a_i is greater (smaller) than the a_{i-1}, the d_i is greater (smaller) than d_{i-1}.
4. All the values are distinct and alternately odd, even.
5. When $a_n = a_j$, $j < n$, there will be a necklace, i.e., when

$$\frac{ba_{j-1} + c}{2^j} = \frac{ba_{n-1} + c}{2^n}. \tag{20}$$

6. The necklace starts with an odd number and ends on the same number. Examples: $\mathscr{R}\{5,1,13\}$ = $\overparen{5,18,9,22,11,24,3,16,1,14,7,20,5}$
7. Number of cycles (= the length of the necklace) = total numbers on the necklace. The length of the necklace for $\mathscr{R}\{5,1,13\}$ is 12, while for $\mathscr{R}\{91,1,13\}$, it is 2.
8. There will be no chain if $a_n = a_0$, i.e., when.

$$a_o = \frac{ba_{n-1} + c}{2^n}.$$

   Example $\mathscr{R}\{5,1,13\}$ = $\overparen{5,9,11,3,1,7,5}$.
9. If a_0 (or b) and c has a highest common factor (HCF) H, all the numbers will be multiples of H. Examples $\mathscr{R}\{15,5,3\}$ and $\mathscr{R}\{m13,1,13\}$, m is an odd integer.
10. It is possible that $\mathscr{R}\{a_0,b,c\}$ may have small or large number of iterations. Examples $\mathscr{R}\{15,5,3\}$ and $\mathscr{R}\{27,3,1\}$.

Special cases

(i) Collatz Iterative Operation (CIO)*: b* = 3, *c* = 1

1. The CND always terminates in a necklace $\overparen{1,4,1}$. Chain may or may not exist.
2. Whenever a_i is greater (smaller) than a_{i-1}, the d_i is greater (smaller) than d_{i-1}.
3. If a (or b) and c has a highest common factor (HCF) H, all the numbers will be multiples of H.

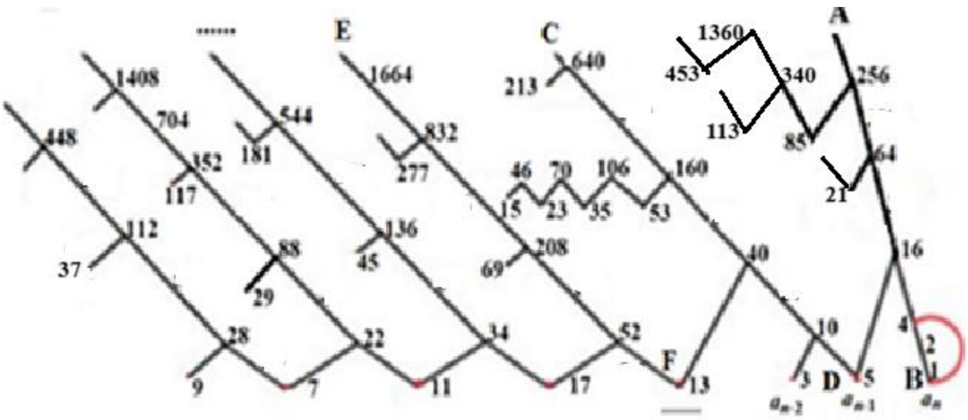


Figure 14: A part of the tree for Collatz Operation

4. Using (18), one can develop the tree in the forward or backward direction. A part of the tree so obtained is shown in Fig. 12. All the numbers on the tree are distinct. Whatever be the value of a_0 (starting point), all paths finally merge into line AB and then terminate in 1.
5. The number from where the journey starts is called a *starting number* (SN). Numbers which form a shape of ⋁ with positive and negative slope lines (7,11,53,85,113) is called a ⋁N, that form a ⋀ (46,70,106) is called ⋀**N** and that forms ⋋ (28,22,34,160) is called a ⋋N. All ⋁Ns are odd, and ⋀Ns and ⋋Ns are even.

A VN can be an odd number either a prime or a compound number. If it is a prime number (9,7,11,13, 113) or a multiple of 3 (3,9,15,21), it can be only SN a_0.

6. The entire NSL will be designated as (VN)-line. Thus, line AB is 1-line, CD is 5-line.
7. On a_i-line the number $d_{i-1} = a_i \times 2^i$, where 2^i is such that results d_{i-1} in an even number. For example, the numbers on the 13-line are 13×even powers of 2….. , the numbers on the 17-line are 17×odd powers of 2.
8. Any positive slope line has only 2 numbers situated at the two ends of the line and forms an odd-even pair. Example 13-40, 21-64.
9. The path starting from a_i and joining the line AB (1-line) at junction j will be designated as $p_{i\text{-}j}$ Thus, the path starting from 9 is $p_{9\text{-}16}$. The path starting from 21 is $p_{21\text{-}64}$.
10. Number of iterations depends on a_0. For example, $\mathscr{R}\{5,3,1\} = 5,16,\widetilde{1,4,1}$ (2 iterations), while $\mathscr{R}\{27,3,1\}$= 27,82,41,124,31,94,47,142,71,204,107,322,161,484,121,364,91,274,137,412,103,310,155,466,233,700,175, 526,263,790,395,1186,593,1780,445,1336,167,502,251,754,377,1132,283,850,425,1276,319,958,479,1438,719, 2158,1079,3238,1619,4858,2429,7288,911,2764,1367,4102,2051,6154,3077,9232,577,1732,433,1300,325,976,61,184,23,70,35,106,53,160,5,16,$\widetilde{1,4,1}$ (43 iterations).
11. A VN meeting the line 1-line is given by
$$r_n = (4r_{n-1} + 1),\, r_0 = 1,\;\; n > 1. \tag{21}$$
Thus, $r_1 = 5$, $r_2 = 21$, $r_3 = 85$, and so on.
12. Similarly, the number q_n incident 5-line is
$$q_n = (4q_{n-1} + 1),\;\; q_o = 3,\;\; n > 1. \tag{22}$$
Thus, $q_1 = 13$, $q_2 = 53$, $q_3 = 213$, and so on.
13. The total number of steps per iteration is 2: One for going up and one for going down. Thus, the total number of steps for n iterations
$$N_s = 2n. \tag{23}$$
If SN is 5, $N_s = 4$, for SN 15, $N_s = 10$, for SN 9, $N_s = 12$.

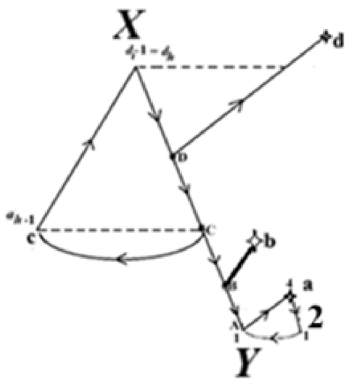


Figure 15: One particular iteration starting from $a_{h\text{-}1}$

14. Refer to Fig. 13. Let d_h be the highest possible value $d_{i\text{-}1}$ corresponding to $a_{h\text{-}1}$. The negative slope line may terminate into a point A, B, C or D.
    (i) If the end point is A (where A = 1), then in the next iteration it will terminate in $\widetilde{1,4,1\}}$.
    (ii) If the end point is B, then in the next cycle it will go to point b which is less than $a_{h\text{-}1}$. Therefore, after a few cycles it will terminate into a necklace $\widetilde{1,4,1\}}$ (see Fig. 12).
    (iii) If the end point is C, then in the next iteration it will go to point c; which is only possible when $\mathscr{R}\{a_0,3,1\}$ come to 1-line and then terminate into the necklace $\widetilde{1,4,1}$ for different values of a_0.
    (iv) If the end point is D, then in the next iteration it will go to a point d which is greater than d_h which does not exist. Therefore, the path DF is not possible.
    Thus, in all cases, it will terminate in $\widetilde{1,4,1}$.

Thus, for all the finite numbers the path terminates into $\overbrace{1,4,1}$.

(ii) **Nollatz function (NF):** $\boldsymbol{b = 1, c = 1}$
This is a special case of CF. Here the gap between a_i and d_i is a minimum. The a_i and d_i will be the consecutive numbers. Thus, all numbers will finally reach 1 and then there will be a necklace $\overbrace{1,2,1}$.
Examples: $\mathscr{R}\{21,1,1\} = 21,22,11,12,3,4,\overbrace{1,2,1}$, $\mathscr{R}\{17,1,1\} = 17,18,9,10,5,6,3,4,\overbrace{1,2,1}$
(iii) **Lagarias Function (LF):** $\boldsymbol{b = 3}$
(A) $\boldsymbol{a_0 \neq c}$**:** It will have both chain and necklace. Examples: $\mathscr{R}\{5,3,13\} = 5,28,7,34,17,64,\overbrace{1,16,1}$.
$\mathscr{R}\{5,3,11\} = 5,26,\overbrace{13,50,25,86,43,35,116,29,98,49,158,79,248,31,104,13}$
For $\mathscr{R}\{x,3,3x\}$, $= x,6x,\overbrace{3x,12x,3x}$. Example: $\mathscr{R}\{3,3,9\} = 3,18,\overbrace{9,36,9}$.
(B) $\boldsymbol{a_0 = c}$*:* CND is $\overbrace{a_0,4a_0,a_0}$, no chain. Example: $\mathscr{R}\{1,3,1\} = \overbrace{1,4,1}$.

(iv) Special case: $\boldsymbol{b = 1}$
(A) $\boldsymbol{a_0 < c}$**:** There will be only necklace. All beads appear in the same cyclic order. All the odd beads will be $< c$. If the number of beads equals c-1, it will be called a perfect necklace (PN). Only restricted c have PN.
Examples: $\mathscr{R}\{5,1,13\} = \overbrace{5,18,9,22,11,24,12,6,3,16,8,4,2,1,14,7,20,10,5}$ is a PN. Here, all the odd numbers $< c$ are included.
$\mathscr{R}\{5,1,17\} = \overbrace{5,22,11,28,14,7,24,12,6,3,20,10,5}$ = S1. Here odd numbers 3,5,7,11 $< c$ are present. If we take ao = 1,3,9,13,15 (odd numbers which are not present in S1),

$\mathscr{R}\{1,1,17\}$ = 1,18,9,26,13,30,15,32,1 = S2. Odd numbers 1,9,13,15 which are not present in S1 are present. We call them *complementary sequences*.
(B) $\boldsymbol{a_0 = mc}$*:* If $a_0 = 13$, then there will be a necklace. All the *odd* beads on the chain will be $\geq c$ and those on the necklace will be $= c$.
$\mathscr{R}\{mc,1,c\}$, (where m is an odd number) $= \ldots,\overbrace{c,2c,c}$.
$\mathscr{R}\{9,1,3\} = 9,12,\overbrace{3,6,3}\}$.
$\mathscr{R}\{33,1,11\} = 33,44,\overbrace{11,22,11}$
(C) $\boldsymbol{a_0 > c}$**:** All the beads on the chain > c, and all the *odd* beads on the necklace < c. Necklace may or may not be a PN.
$\mathscr{R}\{37,1,13\} = 37,50,25,38,19,32,\overbrace{1,14,7,20,5,18,9,22,11,24,3,16,1}$.
$\mathscr{R}\{13,1,9\} = 13,22,11,20,\overbrace{5,14,7,16,1,10,5}$.

(v) Tio function TF: {$\boldsymbol{a_0}$,3, -1}
(A) $\boldsymbol{a_0}$ **positive:** This is a special case of LF. Unlike CF, it does not terminate into $\overbrace{1,4,1}$.
$\mathscr{R}\{7,3,-1\} = \overbrace{7,20,5,14,7}$, $\mathscr{R}\{17,3,-1 = \overbrace{17,50,25,74,37,110,55,164,41,122,61,182,91,272,17}$.
(B) $\boldsymbol{a_0}$ **negative:** In this case all numbers will be negative. $\mathscr{R}\{-a_0,3,-1\} = \mathscr{R}\{-(a_0,3,1)\}$
$\mathscr{R}\{-7,3,-1\} = -7,-22,-11,-34,-17,-52,-13,-40,-5,-16,\overbrace{-1,-4,-1}$
$= -(7,22,11,34,17,52,13,40,5,16,\overbrace{1,4,1})$.

III. CONCLUSION

Some iterative operations on integer numbers have been reviewed. Their properties are summarized. The operations are shown pictorially by chain-necklace diagrams which consists of a chain and a necklace. Iterative operation on RIO-1 ends up in a necklace with either 2 or 5 beads which rotate in cyclic order. TIO-2 have several special cases including Collatz and Nollatz operations. The proof of Collatz's conjecture is given.

REFERENCES

[1] Rathore Tejmal S, Rathore Mradul, Rathore Jayantilal, Khabia Pramila: Some properties of integer numbers: AKGEC Int. J. Technology, 2021,13: 45-50.
[2] Yutaka Nishiyama: The weirdness of number 6174, Int. J. Pure and Applied Mathematics, 2012, 80:363-373.
[3] Rathore Tejmal: Generalized Collatz's conjecture, IETE J. Education, 2023, 64:98-102, DOI: 10.1080/09747338.2023.2178531
[4] Dagnachew Jenber Negash: Collatz Theorem, arXiv:1811.08500v5 [math. GM] 13 Oct 2021.
[5] Honner Patrick: The simple math problem we still can't solve, Sept. 22, 2020.
[6] Guy R K: Don't try to solve these problems, American Math. Monthly, 1983, 90: 35–41, 1983. doi:10.2307/2975688.
[7] Lagarias JC: The Ultimate Challenge: The 3x + 1 Problem, American Mathematical Society, 2010, ISBN 978-0-8218-4940-8. Zbl 1253.11003.