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On the Zeros of Complex Polynomials

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Abstract: *In the framework of the Enestrom-Kakeya Theorem various results have been proved on the location of zeros of complex polynomials. In this paper we give some new results on the zeros of complex polynomials by restricting the real and imaginary parts of their coefficients to certain conitions.*
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I. Introduction and Statement of Results

The following result known as the Enestrom-Kakeya Theorem [10] is well-known in the theory of distribution of zeros of polynomials:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that its coefficients satisfy

$$a_n \ge a_{n-1} \ge ... \ge a_1 \ge a_0 > 0.$$

Then all the zeros of P(z) lie in the closed unit disk $|z| \le 1$.

In the literature [1-9, 12] there exist several generalisations and extensions of this result .
Recently Y. Choo [3] proved the following results:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(a_j)=\beta_j, j=0,1,2,....,n$, such that for some λ and μ, and for some k_1, k_2,

$$k_1\alpha_n \le \alpha_{n-1} \le ... \le \alpha_{\lambda+1} \le \alpha_\lambda \ge \alpha_{\lambda-1} \ge ... \ge \alpha_1 \ge \alpha_0$$
$$k_2\beta_n \le \beta_{n-1} \le ... \le \beta_{\mu+1} \le \beta_\mu \ge \beta_{\mu-1} \ge ... \ge \beta_1 \ge \beta_0.$$

Then P(z) has all its zeros in $R_1 \le |z| \le R_2$ where

$$R_1=\frac{|a_0|}{M_1} \quad \text{and} \quad R_2=\frac{M_2}{|a_n|}$$

with

$$M_1=|a_n|+|(k_1-1)\alpha_n|+|(k_2-1)\beta_n|+2(\alpha_\lambda+\beta_\mu)-(k_1\alpha_n+\beta_n)-(\alpha_0+\beta_0)$$

and

$$M_2=|(k_1-1)\alpha_n|+|(k_2-1)\beta_n|+2(\alpha_\lambda+\beta_\mu)-(k_1\alpha_n+k_2\beta_n)-(\alpha_0+\beta_0)+|a_0|.$$

M. H. Gulzar [8] made an improvement on the above result by proving the following result:

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(a_j)=\beta_j, j=0,1,2,....,n$, such that for some λ and μ, and for some k_1, k_2, τ_1, τ_2,

$$k_1\alpha_n \le \alpha_{n-1} \le ... \le \alpha_{\lambda+1} \le \alpha_\lambda \ge ... \ge \alpha_1 \ge \tau_1\alpha_0$$
$$k_2\beta_n \le \beta_{n-1} \le ... \le \beta_{\mu+1} \le \beta_\mu \ge ... \ge \beta_1 \ge \tau_2\beta_0.$$

Then P(z) has all its zeros in $R_3 \le |z| \le R_4$ where

$$R_3=\frac{|a_0|}{M_3} \quad \text{and} \quad R_4=\frac{M_4}{|a_n|}$$

with

$$M_3 = |a_n| + |(k_1 - 1)\alpha_n| + |(k_2 - 1)\beta_n| + 2(\alpha_\lambda + \beta_\mu) - (k_1\alpha_n + k_2\beta_n)$$
$$+ |(\tau_1 - 1)\alpha_0| - \tau_1\alpha_0 + |(\tau_2 - 1)\beta_0| - \tau_2\beta_0$$

and

$$M_4 = |(k_1 - 1)\alpha_n| + |(k_2 - 1)\beta_n| + 2(\alpha_\lambda + \beta_\mu) - (k_1\alpha_n + k_2\beta_n)$$
$$+ |(\tau_1 - 1)\alpha_0| - \tau_1\alpha_0 + |(\tau_2 - 1)\beta_0| - \tau_2\beta_0 + |a_0|.$$

The aim of this paper is to find a ring-shaped region between two concentric circles with centre not necessarily on the origin. More precisely, we prove the following results:

Theorem 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j) = \alpha_j, \operatorname{Im}(a_j) = \beta_j, j = 0,1,2,....,n$, such that for some λ and μ, and for some k_1, k_2, τ_1, τ_2,

$$k_1\alpha_n \le \alpha_{n-1} \le ... \le \alpha_{\lambda+1} \le \alpha_\lambda \ge ... \ge \alpha_1 \ge \tau_1\alpha_0$$
$$k_2\beta_n \le \beta_{n-1} \le ... \le \beta_{\mu+1} \le \beta_\mu \ge ... \ge \beta_1 \ge \tau_2\beta_0.$$

Then P(z) has all its zeros in $R_5 \le |z - \gamma_1| \le R_6$ where

$$\gamma_1 = \frac{(1-k_1)\alpha_n}{a_n}, \quad R_5 = \frac{|a_0|}{M_5} - \frac{|(k_1 - 1)\alpha_n|}{|a_n|} \quad \text{and} \quad R_6 = \frac{M_6}{|a_n|}$$

with

$$M_5 = |a_n| + |(k_1 - 1)\alpha_n| + |(k_2 - 1)\beta_n| + 2(\alpha_\lambda + \beta_\mu) - (k_1\alpha_n + k_2\beta_n)$$
$$+ |(\tau_1 - 1)\alpha_0| - \tau_1\alpha_0 + |(\tau_2 - 1)\beta_0| - \tau_2\beta_0$$

and

$$M_6 = 2(\alpha_\lambda + \beta_\mu) - (k_1\alpha_n + k_2\beta_n) + |(k_2 - 1)\beta_n|$$
$$+ |(\tau_1 - 1)\alpha_0| - \tau_1\alpha_0 + |(\tau_2 - 1)\beta_0| - \tau_2\beta_0 + |a_0|.$$

Theorem 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j) = \alpha_j, \operatorname{Im}(a_j) = \beta_j, j = 0,1,2,....,n$, such that for some λ and μ, and for some k_1, k_2, τ_1, τ_2,

$$k_1\alpha_n \le \alpha_{n-1} \le ... \le \alpha_{\lambda+1} \le \alpha_\lambda \ge ... \ge \alpha_1 \ge \tau_1\alpha_0$$
$$k_2\beta_n \le \beta_{n-1} \le ... \le \beta_{\mu+1} \le \beta_\mu \ge ... \ge \beta_1 \ge \tau_2\beta_0.$$

Then P(z) has all its zeros in $R_7 \le |z - \gamma_2| \le R_8$ where

$$\gamma_2 = \frac{i(1-k_2)\beta_n}{a_n}, \quad R_7 = \frac{|a_0|}{M_7} - \frac{|(k_2 - 1)\beta_n|}{|a_n|} \quad \text{and} \quad R_8 = \frac{M_8}{|a_n|}$$

with

$$M_7 = |a_n| + |(k_1 - 1)\alpha_n| + |(k_2 - 1)\beta_n| + 2(\alpha_\lambda + \beta_\mu) - (k_1\alpha_n + k_2\beta_n)$$
$$+ |(\tau_1 - 1)\alpha_0| - \tau_1\alpha_0 + |(\tau_2 - 1)\beta_0| - \tau_2\beta_0$$

and

$$M_8 = 2(\alpha_\lambda + \beta_\mu) - (k_1\alpha_n + k_2\beta_n) + |(k_1 - 1)\alpha_n|$$
$$+ |(\tau_1 - 1)\alpha_0| - \tau_1\alpha_0 + |(\tau_2 - 1)\beta_0| - \tau_2\beta_0 + |a_0|.$$

Remark 1: The bounds for the zeros of P(z) in both Theorem 1 and Theorem 2 are easily seen to be sharper than those of Theorems B and C. For different values of the parameters k_1, k_2, τ_1, τ_2, we get many other interesting results. For example for

$k_1 = 1,\ \tau_2 = 1$, in Theorem 2 ,we get a result due to B. L. Raina et al [11].

For $\tau_1 = 1, \tau_2 = 1$, Theorem 1 reduces to Theorem B.

For $k_1 = 1,\ k_2 = 1$,, in Theorem 1, we get the following result:

Corollary 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j, \mathrm{Im}(a_j) = \beta_j,\ j = 0,1,2,....,n$, such that for some λ and μ, and for some τ_1, τ_2,

$$\alpha_n \le \alpha_{n-1} \le ... \le \alpha_{\lambda+1} \le \alpha_\lambda \ge ... \ge \alpha_1 \ge \tau_1\alpha_0$$

$$\beta_n \le \beta_{n-1} \le ... \le \beta_{\mu+1} \le \beta_\mu \ge ... \ge \beta_1 \ge \tau_2\beta_0.$$

Then P(z) has all its zeros in $R_9 \le |z| \le R_{10}$ where

$$R_9 = \frac{|a_0|}{M_9} \quad \text{and} \quad R_{10} = \frac{M_{10}}{|a_n|}$$

with

$$M_9 = |a_n| + 2(\alpha_\lambda + \beta_\mu) - (\alpha_n + \beta_n) + |(\tau_1 - 1)\alpha_0| - \tau_1\alpha_0 + |(\tau_2 - 1)\beta_0| - \tau_2\beta_0$$

and

$$M_{10} = 2(\alpha_\lambda + \beta_\mu) - (\alpha_n + \beta_n) + |(\tau_1 - 1)\alpha_0| - \tau_1\alpha_0 + |(\tau_2 - 1)\beta_0| - \tau_2\beta_0 + |a_0| \ .$$

II. Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... \\
&\quad + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} + (k_1\alpha_n - \alpha_{n-1})z^n - (k_1 - 1)\alpha_n z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + ... \\
&\quad + (\alpha_1 - \tau_1\alpha_0)z + (\tau_1 - 1)\alpha_0 z + a_0 + i\{(k_2\beta_n - \beta_{n-1})z^n - (k_2 - 1)\beta_n z^n \\
&\quad + (\beta_{n-1} - \beta_{n-2})z^{n-1} + ... + (\beta_1 - \tau_2\beta_0)z + (\tau_2 - 1)\beta_0 z\} \\
&= -z^n[\{a_n z + (k_1 - 1)\alpha_n\} - \{(k_1\alpha_n - \alpha_{n-1}) + \frac{\alpha_{n-1} - \alpha_{n-2}}{z} + ... \\
&\quad + \frac{\alpha_1 - \tau_1\alpha_0}{z^{n-1}} + \frac{(\tau_1 - 1)\alpha_0}{z^{n-1}} + \frac{a_0}{z^n}\} - i\{(k_2\beta_n - \beta_{n-1}) - \frac{(k_2 - 1)\beta_n}{z} \\
&\quad + \frac{\beta_{n-1} - \beta_{n-2}}{z} + ... + \frac{\beta_1 - \tau_2\beta_0}{z^{n-1}} + \frac{(\tau_2 - 1)\beta_0}{z^n}\}]
\end{aligned}$$

For $|z| > 1$, we have $\frac{1}{|z|^{n-j}} < 1,\ j = 0,1,......,n$ and, therefore ,

$$|F(z)| \ge |-z|^n[|a_n z + (k_1 - 1)\alpha_n| - \{|k_1\alpha_n - \alpha_{n-1}| + \frac{|\alpha_{n-1} - \alpha_{n-2}|}{|z|} + ...$$

$$+\frac{|\alpha_1-\tau_1\alpha_0|}{|z|^{n-1}}+\frac{|(\tau_1-1)\alpha_0|}{|z|^{n-1}}+\frac{|a_0|}{|z|^n}+|k_2\beta_n-\beta_{n-1}|+\frac{|(k_2-1)\beta_n|}{|z|}$$

$$+\frac{|\beta_{n-1}-\beta_{n-2}|}{|z|}+......+\frac{|\beta_1-\tau_2\beta_0|}{|z|^{n-1}}+\frac{|(\tau_2-1)\beta_0|}{|z|^n}\}]$$

$$>|z|^n[|a_nz+(k_1-1)\alpha_n|-\{|k_1\alpha_n-\alpha_{n-1}|+|\alpha_{n-1}-\alpha_{n-2}|+......+|\alpha_1-\tau_1\alpha_0|$$

$$+|(\tau_1-1)\alpha_0|+|a_0|+|k_2\beta_n-\beta_{n-1}|+|(k_2-1)\beta_n|+|\beta_{n-1}-\beta_{n-2}|$$

$$+|\beta_1-\tau_2\beta_0|+|(\tau_2-1)\beta_0|\}]$$

$$=|z|^n[|a_nz+(k_1-1)\alpha_n|-\{(\alpha_{n-1}-k_1\alpha_n)+(\alpha_{n-2}-\alpha_{n-1})+......+(\alpha_\lambda-\alpha_{\lambda+1})$$

$$+(\alpha_\lambda-\alpha_{\lambda-1})+......+(\alpha_1-\tau_1\alpha_0)+|(\tau_1-1)\alpha_0|+|a_0|+(\beta_{n-1}-k_2\beta_n)$$

$$+|(k_2-1)\beta_n|+(\beta_{n-2}-\beta_{n-1})+......+(\beta_\mu-\beta_{\mu+1})+(\beta_\mu-\beta_{\mu-1})+......$$

$$+(\beta_1-\tau_2\beta_0)+|(\tau_2-1)\beta_0|\}]$$

$$=|z|^n[|a_nz+(k_1-1)\alpha_n|-\{2(\alpha_\lambda+\beta_\mu)-(k_1\alpha_n+k_2\beta_n)+|(k_2-1)\beta_n|$$

$$+|(\tau_1-1)\alpha_0|+|(\tau_2-1)\beta_0|-\tau_1\alpha_0-\tau_2\beta_0+|a_0|\}]$$

$$>0$$

if

$$|a_nz+(k_1-1)\alpha_n|>\{2(\alpha_\lambda+\beta_\mu)-(k_1\alpha_n+k_2\beta_n)+|(k_2-1)\beta_n|$$

$$+|(\tau_1-1)\alpha_0|+|(\tau_2-1)\beta_0|-\tau_1\alpha_0-\tau_2\beta_0+|a_0|\}$$

or

$$\left|z+\frac{(k_1-1)\alpha_n}{a_n}\right|>\frac{1}{|a_n|}[\{2(\alpha_\lambda+\beta_\mu)-(k_1\alpha_n+k_2\beta_n)+|(k_2-1)\beta_n|$$

This shows that those zeros of F(z) and hence P(z) whose modulus is greater than 1 lie in

$$\left|z+\frac{(k_1-1)\alpha_n}{a_n}\right|\le\frac{1}{|a_n|}[\{2(\alpha_\lambda+\beta_\mu)-(k_1\alpha_n+k_2\beta_n)+|(k_2-1)\beta_n|$$

$$+|(\tau_1-1)\alpha_0|+|(\tau_2-1)\beta_0|-\tau_1\alpha_0-\tau_2\beta_0\}].$$

Since the zeros of P(z) of modulus less than or equal to 1 already satisfy the above inequality, it follows that all the zeros of P(z) lie in

$$|z-\gamma_1|\le R_6.$$

To prove the other half of the theorem, we have

$$F(z)=-a_nz^{n+1}+(k_1\alpha_n-\alpha_{n-1})z^n-(k_1-1)\alpha_nz^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+......$$

$$+(\alpha_1-\tau_1\alpha_0)z+(\tau_1-1)\alpha_0z+i\{(k_2\beta_n-\beta_{n-1})z^n-(k_2-1)\beta_nz^n$$

$$+(\beta_{n-1}-\beta_{n-2})z^{n-1}+......+(\beta_1-\tau_2\beta_0)z+(\tau_2-1)\beta_0z\}+a_0$$

$$=Q(z)+a_0,$$

where

$$Q(z)=-a_nz^{n+1}+(k_1\alpha_n-\alpha_{n-1})z^n-(k_1-1)\alpha_nz^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+......$$

$$+(\alpha_1-\tau_1\alpha_0)z+(\tau_1-1)\alpha_0 z+i\{(k_2\beta_n-\beta_{n-1})z^n-(k_2-1)\beta_n z^n$$
$$+(\beta_{n-1}-\beta_{n-2})z^{n-1}+......+(\beta_1-\tau_2\beta_0)z+(\tau_2-1)\beta_0 z\}$$

For $|z|\le 1$,

$$|Q(z)|\le |a_n z+(k_1-1)\alpha_n|+(\alpha_{n-1}-k_1\alpha_n)+(\alpha_{n-2}-\alpha_{n-1})+......+(\alpha_\lambda-\alpha_{\lambda+1})+(\alpha_\lambda-\alpha_{\lambda-1})$$

$$+......+(\alpha_1-\tau_1\alpha_0)+|(\tau_1-1)\alpha_0|+|(k_2-1)\beta_n|+(\beta_{n-1}-k_2\beta_n)+(\beta_{n-2}-\beta_{n-1})$$
$$+......+(\beta_\mu-\beta_{\mu+1})+(\beta_\mu-\beta_{\mu-1})+......+(\beta_1-\tau_2\beta_0)+|(\tau_2-1)\beta_0|$$
$$\le |a_n z+(k_1-1)\alpha_n|+2(\alpha_\lambda+\beta_\mu)-(k_1\alpha_n+k_2\beta_n)+|(k_2-1)\beta_n|$$
$$+|(\tau_1-1)\alpha_0|+|(\tau_2-1)\beta_0|-\tau_1\alpha_0-\tau_2\beta_0 .$$
$$=|a_n z+(k_1-1)\alpha_n|+R$$
$$\le |a_n|+|(k_1-1)\alpha_n|+R=M_5 ,$$

where

$$R=2(\alpha_\lambda+\beta_\mu)-(k_1\alpha_n+k_2\beta_n)+|(k_2-1)\beta_n|$$
$$+|(\tau_1-1)\alpha_0|-\tau_1\alpha_0+|(\tau_2-1)\beta_0|-\tau_2\beta_0$$

Since Q(z) is analytic and Q(0)=0, it follows by Rouche's theorem that

$$|Q(z)|\le M_5|z| \quad \text{for } |z|\le 1.$$

Therefore , for $|z|\le 1$,

$$|F(z)|=|a_0+Q(z)|$$
$$\ge |a_0|-|Q(z)|$$
$$\ge |a_0|-M_5|z|$$
$$>0$$

if

$$|z|<\frac{|a_0|}{M_5}.$$

This shows that F(z) does not vanish in $|z|<\frac{|a_0|}{M_5}$.It is easy to see that $M_5\le |a_0|$. In other words, it follows that F(z) and hence P(z) has all its zeros in $\frac{|a_0|}{M_5}\le |z|$.

Since

$$|z|=|z-\gamma_1+\gamma_1|\le |z-\gamma_1|+|\gamma_1| ,$$

we have

$$\frac{|a_0|}{M_5}\le |z|\le |z-\gamma_1|+|\gamma_1| .$$

Therefore

$$\frac{|a_0|}{M_5}-|\gamma_1|\le |z-\gamma_1|$$

i.e.

$$\frac{|a_0|}{M_5} - \left|\frac{(k_1 - 1)\alpha_n}{a_n}\right| \le |z - \gamma_1|.$$

Hence, it follows that P(z) has all its zeros in

$$\frac{|a_0|}{M_5} - \left|\frac{(k_1 - 1)\alpha_n}{a_n}\right| \le |z - \gamma_1|.$$

That completes the proof of Theorem 1.

Proof of Theorem 2: Similar to that of Theorem 2.

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