International Journal of Engineering Inventions
e-ISSN: 2278-7461, p-ISSN: 2319-6491
Volume 2, Issue 12 (August 2013) PP: 37-42

Adomain Decomposition Method for ∅4 Klein Gordon Equation

Fadhil H. Easif[1], Saad A. Manaa[2], Dilveen Mekaeel[3]
[1,2]*Department of Mathematics, Faculty of Science, University of Zakho, Duhok, Kurdistan Region, Iraq*
[3]*Department of Mathematics, College of Science, University of Duhok, Duhok, Kurdistan Region, Iraq*

Abstract: *$\emptyset^4$ Klein Gordon Equation has been solved numerically by using two methods: finite difference method (FDM) and Adomain decomposition method (ADM) and we discover that the ADM is much more accurate than FDM in this kind of models as shown in the example(1,2).*
Keywords: *Klein Gordon Equation, ADM method, FDM method.*

I. INTRODUCTION

A finite difference method proceeds by replacing the derivatives in the differential equations by finite difference approximations. We assume the function whose derivatives are to be approximated by using Taylor's expansion. This gives a large algebraic system of equations to be solved in place of the differential equation, something that is easily solved on a computer [4].

The main advantage of the method is that it can provide analytical or an approximated solution to a rather wide class of nonlinear (and stochastic) equations without linearization, perturbation, closure approximation, or discretization methods. Unlike the common methods which are only applicable to systems with weak nonlinearity and small perturbation and may change the physics of the problem due to simplification, ADM gives the approximated solution of the problem without any simplification. Another important advantage is that the method is capable of greatly reducing the size of computation work while still maintaining high accuracy of the numerical solution [1,2].

I.1. MATHEMATICAL MODEL:

The Klein-Gordon equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u - f(u) \qquad (1)$$

was named after the physicists Oskar Klein and Walter Gordon, who in 1926 proposed that it describes relativistic electrons. Some other authors make the similar claims in the same year. The equation was first considered as a quantum wave equation by Schrödinger in his search for an equation describing de Broglie waves. The equation is found in his notebooks from late 1925, before he made the discovery of the equation that now bears his name. He rejected it because he couldn't make it fit data (the equation doesn't take in account the spin of the electron), the way he found his equation was by making simplifications in the Klein Gordon equation. In 1927, soon after the Schrödinger equation was introduced, Vladimir Fock wrote an article about its generalization for the case of magnetic fields, where forces were dependent on velocity, and independently derived this equation. Both Klein and Fock used Kaluza and Klein's method [6].

When$f(u) = \sin(u)$, then equation (1) becomes sine-Gordon equation, which is found by Zabusky and kruskal in 1965.

Fiore et al. (2005) gave arguments for the existence of exact travelling wave solutions of a perturbed sine Gordon equation on the real line or on the circle and classified them [7].

When $f(u) = mu - \varepsilon u^3$then equation (1) is called $\emptyset^4$–nonlinear klein Gordon equation ($\emptyset^4$ equation)[8]

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - mu + \varepsilon u^2 \qquad (2)$$

or [33]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - mu + \varepsilon u^2 \qquad (3)$$

With initial and boundary conditions [10]

$u(x, 0) = f(x)$ and $\frac{\partial u(x,0)}{\partial t} = 0 \quad , t \geq 0 \ , 0 < x < 2\pi$

$u(0, t) = u(2\pi, t) = 0$

Equation (3) arises in quantum field theory with m denoting mass and ε is
coupling constant [11].

The φ^4 equation has become an important subject because of its numerous applications in condensed matter physics. It describes, for example, structural phase transitions in ferroelectric and ferromagnetic materials, topological excitations in quasi one dimensional system like biological macromolecules and hydrogen chains, or polymers, etc. Its simplest localized solutions so-called "kinks" which are related to the motion of the aforementioned topological excitations, e.g., domain walls in second order phase transitions, or polymerization mismatches. A more realistic modeling of physical situation in condensed matter physics often requires the inclusion of perturbations of different types like thermal noise and time or spatial dependent potential fluctuations[12].

II. MATERIALS AND METHODS

II.1 Finite Difference Method

Our aim is to approximate solutions to differential equations (i.e., to find a function or some discrete approximation to this function) which satisfies a given relationship between various of its derivatives on some given region of space and/or time, along with some boundary conditions along the edges of this domain. Now, the formulation of this technique gives:

$$u'(x) = \frac{u(x+h)-u(x)}{h}$$

$$u''(x) = \frac{1}{h^2}\big(u(x+h) - 2u(x) + u(x-h)\big)$$

Let the coordinate (x, t) of the grid points be

$$x = ph, t = qk$$

Where p, q are integers.

Denote the values of u at these mesh points by

$$u(ph, qk) = u_{p,q}$$

We start by partitioning the rectangle $R = \{(x,t): 0 \le x \le 2\pi, 0 \le t \le b\}$ into a grid consisting of $n-1$ by $m-1$ rectangle with sides $\Delta x = h$ and$\Delta t = k$, as shown in Figure (1) [4]:

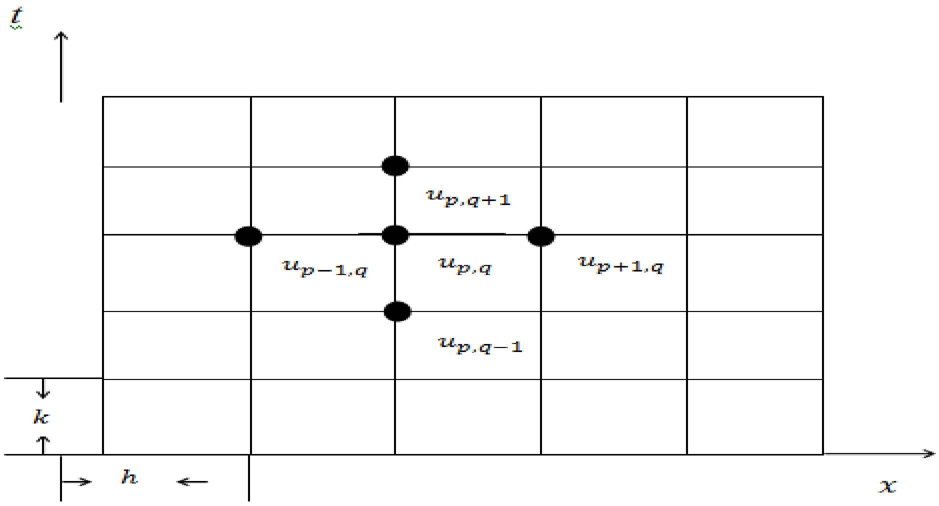

Figure (1): The mesh of finite difference method

We shall use a difference-equation method to calculate approximation

$$\{u_{p,q} : p = 1,2,3, \dots, n\} \quad , q = 1,2,3,4, \dots, m$$

Now, equation (3) be as follows:

$$\frac{u_{p,q+1} - 2u_{p,q} + u_{p,q-1}}{k^2} = \frac{u_{p+1,q} - 2u_{p,q} + u_{p-1,q}}{h^2} - mu_{p,q} + \varepsilon u_{p,q}^3$$

$$u_{(p,q+1)} - 2u_{p,q} + u_{p,q-1} = \frac{k^2}{h^2}\big(u_{(p+1,q)} - 2u_{p,q} + u_{p-1,q}\big) - mk^2 u_{p,q} + \varepsilon k^2 u_{p,q}^3$$

setting $r = \frac{k^2}{h^2}$

$$u_{p,q+1} - 2u_{p,q} + u_{p,q-1} = r\big(u_{(p+1,q)} - 2u_{p,q} + u_{p-1,q}\big) - mk^2 u_{p,q} + \varepsilon k^2 u_{p,q}^3$$

$$u_{p,q+1} = r\big(u_{(p+1,q)} + u_{p-1,q}\big) - 2ru_{p,q} + 2u_{p,q} - u_{p,q-1} - mk^2 u_{p,q} + \varepsilon k^2 u_{p,q}^3$$

$$u_{p,q+1} = r\big(u_{(p+1,q)} + u_{p-1,q}\big) + (2 - 2r - mk^2 + \varepsilon k^2 u_{p,q}^2)u_{p,q} - u_{p,q-1} \quad (4)$$

Equation (4) represents the explicit difference approximation for $\emptyset^4$ equation, which is used to find the row $(q+1)$ across the grid where the approximation in both rows (q), $(q-1)$ are known. The four known values on the right hand side $u_{(p,q-1)}, u_{(p-1,q)}, u_{(p+1,q)}$ and $u_{(p,q)}$ are used to create the approximation $u_{(p,q+1)}$. In order to use formula (4) to compute the third row, the first two starting rows of values corresponding to $q = 1$ and $q = 2$ must be supplied. The boundary function $f(x_p, t_1) = f(x_p)$, $p = 1,2,3, \dots \dots .. n$ is used to help produce the first row. Since the second row cannot be determined in a usually way, we may use the Taylor's formula $u(x,t)$ at $x = x_p$ to compute it.

$u_{p,2} = u_{p,1} + \frac{r}{2}(u_{p+1,1}+u_{p-1,1}) - ru_{p,1} - \frac{k^2}{2}mu_{p,1} + \frac{k^2}{2}\varepsilon u_{p,1}^3$ (5)

II.1 Adomain Decomposition Method

We will present a review of the standard Adomin decomposition method [1,2,3]. To achieve our goal we consider the general form of differential equation:

$Lu + Ru + Nu = g(x)$

Where L is the highest order derivative which is assumed to be easily invertible R is the remainder of the linear operator, Nu represents the nonlinear terms, and g is the source term.

The Adomin's technique consists of approximating the solution an infinite series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

And decomposing the nonlinear operator N as

$$N(u) = \sum_{n=0}^{\infty} A_n$$

Where A_n is Adomian's polynomial of $u_0, u_1, u_2, u_3 \ldots\ldots, u_n$

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty}\lambda^i u_i)]_{\lambda=0}, n \geq 0$$

and we consider the nonlinear homogenous Klein-Gordon $\emptyset^4$equation:

$$u_{tt} = u_{xx} + mu - \varepsilon u^3$$

Applying the decomposition method (see [1], [2], [11])

$$L_{tt}u = L_{xx}u + mu - \varepsilon u^3$$

Where $L_{tt} = \frac{\partial^2}{\partial t^2}$ and $L_{xx} = \frac{\partial^2}{\partial x^2}$

Assuming the inverse of operator L_{tt} exist, it can be taken as

$$L_{tt}^{-1} = \int_0^t\int_0^t(.)\ dt\,dt.$$

Therefore, applying on both sides with L_{tt}^{-1} yields

$$u(x,t) = u(x,0) + tu^{'}(x,0) + L_{tt}^{-1}(L_{xx}u) + mL_{tt}^{-1}u - \varepsilon L_{tt}^{-1}u^3$$

By using initial condition we get

$$u(x,t) = u_0(x) + tu_0^{'}(x) + L_{tt}^{-1}(L_{xx}u) + mL_{tt}^{-1}u - \varepsilon L_{tt}^{-1}u^3$$

So, we get:

$$\sum_{n=0}^{\infty} u(t,x) = u_0(x) + tu_0^{'}(x) + L_{tt}^{-1}(L_{xx}\sum_{n=0}^{\infty} u) + mL_{tt}^{-1}\sum_{n=0}^{\infty} u - \varepsilon L_{tt}^{-1}\sum_{n=0}^{\infty} A_n$$

Then we have:

$u_0 = u_0(x)$,

and

$$u_{k+1} = L_{tt}^{-1}(L_{xx}u_k) + mL_{tt}^{-1}u_k - \varepsilon L_{tt}^{-1}A_k$$

Where $k \geq 0$.

For $k = 0$,

$A_0 = \frac{1}{0!}\frac{d^0}{d\lambda^0}[F(\sum_{i=0}^{0}\lambda^0 u_0)]_{\lambda=0} = F(u_0) = u_0^3$

$A_0 = u_0^3$

Then we get ,

$u_1 = L_{tt}^{-1}(L_{xx}u_0) + mL_{tt}^{-1}u_0 - \varepsilon L_{tt}^{-1}u_0^3 = L_t^{-1}(L_{xx}u_0)t + mL_t^{-1}u_0 t - \varepsilon L_t^{-1}u_0^3 t$

$= ((L_{xx}u_0) + mu_0 - \varepsilon u_0^3)\frac{t^2}{2} = (L_{xx}u_0)\frac{t^2}{2} + mu_0\frac{t^2}{2} - \varepsilon u_0^3\frac{t^2}{2}$,

Let $D1 = ((L_{xx}u_0) + mu_0 - \varepsilon u_0^3)$

Then $u_1 = D1\frac{t^2}{2}$

For $k = 1$

$A_1 = \frac{1}{1!}\frac{d^1}{d\lambda^1}[F(\sum_{i=0}^{1}\lambda^i u_i)]_{\lambda=0} = \frac{d}{d\lambda}[F(u_0 + \lambda^1 u_1)]_{\lambda=0} = \frac{d}{d\lambda}[(u_0 + \lambda^1 u_1)^3]_{\lambda=0}[3(u_0 + \lambda^1 u_1)^2 u_1]_{\lambda=0} = 3u_0^2 u_1$

we get:

$u_2 = L_{tt}^{-1}(L_{xx}u_1) + mL_{tt}^{-1}u_1 - \varepsilon L_{tt}^{-1}\,3u_0^2 u_1 = L_t^{-1}(L_{xx}u_1)t + mL_t^{-1}u_1 t - 3\varepsilon L_t^{-1}u_0^2 u_1 t$

$= (L_{xx}u_1)\frac{t^2}{2} + mu_1\frac{t^2}{2} - 3\varepsilon u_0^2 u_1\frac{t^2}{2} = (L_{xx}u_1 + mu_1 - 3\varepsilon u_0^2 u_1)\frac{t^2}{2}$

Let, D2=$(L_{xx}u_1 + mu_1 - 3\varepsilon u_0^2 u_1)$

Then $u_2 = \mathrm{D2}\frac{t^2}{2}$

For $k = 2$

$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}[F(\sum_{i=0}^{2}\lambda^i u_i)]_{\lambda=0} = \frac{1}{2}\frac{d^2}{d\lambda^2}[(u_0 + \lambda^1 u_1 + \lambda^2 u_2)^3]_{\lambda=0} = \frac{3}{2}\frac{d}{d\lambda}[(u_0 + \lambda^1 u_1 + \lambda^2 u_2)^2(u_1 + 2\lambda u_2)]_{\lambda=0}$

$$=\frac{3}{2}[(u_0+\lambda^1u_1+\lambda^2u_2)^2(2u_2)+2(u_0+\lambda^1u_1+\lambda^2u_2)(u_1+2\lambda u_2)(u_1+2\lambda u_2)]_{\lambda=0}$$
$$=3u_0^2u_2+3u_0u_1^2$$

We get:

$$u_3=L_{tt}^{-1}(L_{xx}u_2)+mL_{tt}^{-1}u_2-\varepsilon L_{tt}^{-1}(3u_0^2u_2+3u_0u_1^2)=L_t^{-1}(L_{xx}u_2)t+mL_t^{-1}u_2t-\varepsilon L_t^{-1}(3u_0^2u_2+3u_0u_1^2)t\,,$$

$$=L_{xx}(u_2)\frac{t^2}{2}+mu_2\frac{t^2}{2}-\varepsilon(3u_0^2u_2+3u_0u_1^2)\frac{t^2}{2}\,D3=L_{xx}(u_2)+mu_2-\varepsilon(3u_0^2u_2+3u_0u_1^2),$$
$$u_3=D3\frac{t^2}{2},$$

and so on for $k=3,4,\ldots$

III. APPLICATION (NUMERICAL EXAMPLES)

We solve the following examples numerically to illustrate the efficiency of the presented methods, suppose we have the system

Example 1*:* [6]

$$\frac{\partial^2u}{\partial t^2}=-\alpha\frac{\partial^2u}{\partial x^2}-\beta u-\gamma u^3 \qquad ,t>0$$

With the initial conditions

$$u(x,0)=Btan(Kx),u_t=BcK\,sec^2(Kx) \qquad ,-1\le x\le 1$$

We take $\alpha=-2.5,\ \beta=1,\ \gamma=1.5.$

Where $B=\sqrt{\frac{\beta}{\gamma}}$ and $K=\sqrt{\frac{-\beta}{2(\alpha+c^2)}}$

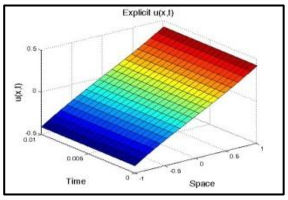

Fig. (2) ADM solution to 0<t<0.01 and -1<x<1 with dx=0.105263157894737

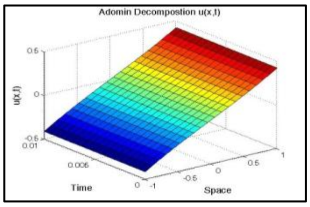

Fig. (3) EFDM solution to 0<t<0.01 and -1<x<1 with dx=0.105263157894737

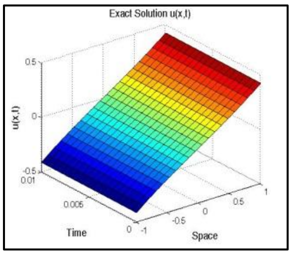

Fig.(4) Exact solution to 0<t<0.01 and -1<x<1 with dx=0.105263157894737

Table (1) comparison between ADM, EFDM and Exact solutions at t=2 with dx=0.020408163265306

EXACT	ADM	EFDM
-0.414411214133078	-0.414411214101644	-0.414411214133078
-0.364670241057009	-0.364670241031911	-0.364669012962060
-0.317087953610942	-0.317087953590899	-0.317086928637545
-0.271306324156148	-0.271306324140212	-0.271305477633181
-0.22701112796834	-0.227011127955811	-0.227010440389676
-0.183922496540239	-0.183922496530586	-0.183921952741587
-0.141787250668732	-0.141787250661578	-0.141786839088029
-0.100372541039790	-0.100372541034864	-0.100372253179315
-0.059460438539449	-0.059460438536567	-0.059460268585494
-0.018843195109709	-0.018843195108769	-0.018843139679920

0.021681051104011	0.021681051103044	0.021680993104744
0.062312318757989	0.062312318755080	0.062312146183729
0.103252748218675	0.103252748213720	0.103252457634093
0.144710656400331	0.144710656393147	0.144710241934291
0.186904856153030	0.186904856143345	0.186904309243295
0.230069434484844	0.230069434472276	0.230068743494559
0.274459218894519	0.274459218878543	0.274458368570347
0.320356215567061	0.320356215546973	0.320355186294337
0.368077383902961	0.368077383877811	0.368076150876117
0.417984229397421	0.417984229365926	0.417984229397421

Example 2*:* [see (6)]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - u - u^3 \qquad ,t > 0$$

With the initial conditions

$$u(x,0) = A[1 + \cos(\frac{2\pi x}{L})],\ u_t = 0 \qquad ,0 \le x \le L$$

The boundary conditions are given by

$u_x(0,t) = 0,\quad u_x(L,t) = 0$, where L=1 and A=1.5

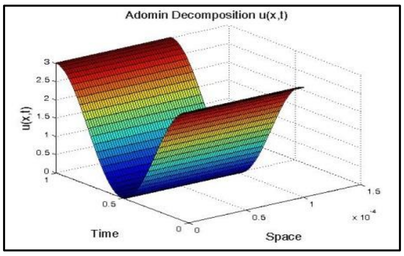

Fig. (5)) ADM solution to 0<t<0.0001 and 0<x<1 with dx=0.020408163265306

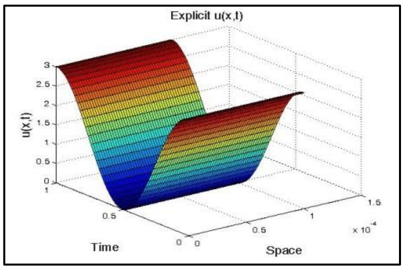

Fig. (6) EFDM solution to 0<t<0.0001 and 0<x<1 with dx=0.020408163265306

Table (2) comparison between ADM, EFDM solution at t=2 with dx=0.020408163265306

ADM	EFDM
2.987685020734869	2.987685020550971
2.950942294558544	2.950942294379764
2.806978056185084	2.806978056025879
2.577524025146591	2.577524025017099
2.435234702788101	2.435234702676172
2.107175014683591	2.107175014609870
1.739399842550069	1.739399842515833
1.548077366357483	1.548077366342586
1.355965461138478	1.355965461142287
1.166218599065529	1.166218599087200
0.981952418368039	0.981952418406554
0.806192564638747	0.806192564692939
0.641825009816746	0.641825009885315
0.491548664608025	0.491548664689548
0.357831062446298	0.357831062539230
0.242867842662239	0.242867842764914
0.148546698146371	0.148546698257009
0.076416379483997	0.076416379600713

0.027661264513402	0.027661264634223
0.003081910874495	0.003081910997386
0.003081910874495	0.003081910997386
0.027661264513402	0.027661264634223
0.076416379483997	0.076416379600713
0.148546698146371	0.148546698257009
0.242867842662239	0.242867842764914
0.357831062446298	0.357831062539229
0.491548664608024	0.491548664689547
0.641825009816745	0.641825009885314
0.806192564638747	0.806192564692939
0.981952418368038	0.981952418406553
1.166218599065527	1.166218599087198
1.355965461138476	1.355965461142286
1.548077366357482	1.548077366342585
1.739399842550069	1.739399842515833
1.926791379946547	1.926791379892598
2.107175014683589	2.107175014609868

IV. CONCLUSION

We saw that Adomain decomposition method is much more accurate than finite difference method for solving □^{4} Klein Gordon Equation and this kind of models as shown in figures (2-4) and table (1) for example (1) and figures (5-6) and table (2).

REFERENCE

[1]. Adomian G., Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston, MA, (1994).

[2]. Adomian G., A review of the decomposition method in applied mathematics, J. Math. Anal. Appl. 135 (1988) 501–544.

[3]. Adomian G., R.Rach, Analytic solution of nonlinear boundary-value problems in several dimensions by decomposition, J. Math. Anal. Appl. 174 (1993) 118-137.

[4]. Chen Liang, Xu Yong, Liu Zhongfei and Han Jiahua, The extended jacobian elliptic function expansion method and its application to nonlinear wave equation, Fizika A 12 (4) (2003) 161-170.

[5]. Fiore, G., A. D'anna and M. De Angelis, Towards soliton solutions of a perturbed sine gordon equation, arXiv:mathPh/0507005 v1, (2005) 1-16.

[6]. Li Q. , Z. Zheng and H. Liu, numerical solution of nonlinear klein-gordon equation using lattice boltzmann method, J. Applied Mathematics, 2 (2011) 1479-1485.

[7]. Manna, M.A. and V. Merle, asymptotic dynamics of short-waves in nonlinear dispersive models, arXiv: solvint/9710013 v1, (1997) 1-4.

[8]. Mathews J .H and K.D. Fink, Numerical Method Using Matlab, Prentice-Hall, Inc. (2004).

[9]. Ole H. Hald, Optimal prediction and the klein-gordon equation, arXiv:math.NA/9910048 v1 (1999) 1-16.

[10]. Robert C. Mcowen, Partial Differential Equations: Methods and Applications, Library of Congress Cataloging (1995).

[11]. Saad A. Manaa, Some numerical methods for Schnackenberg model, International Journal of Engineering Inventions, 2 (2) (2013) 67-74: e-ISSN: 2278-7461.

[12]. Webpage: http://en.wikipedia.org/wiki/Klein-Gordon_equation

[13]. Yuri S. Kivshar, Angel Sanchez and Luis Vazquez; "Kink Decay in a parametrically Driven φ^4 Chin, Phys.Rev.A, 45 (2) (1992) 1207-1212.