International Journal of Engineering Inventions
e-ISSN: 2278-7461, **p**-ISBN: 2319-6491
Volume 2, Issue 3 (February 2013) PP: 49-58

Further Generalizations of Enestrom-Kakeya Theorem

M. H. Gulzar
Department of Mathematics, University of Kashmir, Srinagar 190006

Abstract:- Many generalizations of the Enestrom –Kakeya Theorem are available in the literature. In this paper we prove some results which further generalize some known results.
Mathematics Subject Classification: 30C10,30C15

Keywords and phrases:- Complex number, Polynomial , Zero, Enestrom – Kakeya Theorem.

I. INTRODUCTION AND STATEMENT OF RESULTS

The Enestrom –Kakeya Theorem (see[6]) is well known in the theory of the distribution of zeros of polynomials and is often stated as follows:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n whose coefficients satisfy

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0 .$$

Then P(z) has all its zeros in the closed unit disk $|z| \le 1$.

In the literature there exist several generalizations and extensions of this result. Joyal et al [5] extended it to polynomials with general monotonic coefficients and proved the following result:

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n whose coefficients satisfy

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 .$$

Then P(z) has all its zeros in

$$|z| \le \frac{a_n - a_0 + |a_0|}{|a_n|} .$$

Aziz and zargar [1] generalized the result of Joyal et al [6] as follows:

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \ge 1$

$$ka_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 .$$

Then P(z) has all its zeros in

$$|z+k-1| \le \frac{ka_n - a_0 + |a_0|}{|a_n|} .$$

For polynomials ,whose coefficients are not necessarily real, Govil and Rahman [2] proved the following generalization of Theorem A:

Theorem C: If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$ and $\operatorname{Im}(a_j)=\beta_j$, j=0,1,2,……,n, such that

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \alpha_0 \ge 0 ,$$

where $\alpha_n > 0$, then P(z) has all its zeros in

$$|z| \le 1 + \left(\frac{2}{\alpha_n}\right)\left(\sum_{j=0}^{n} |\beta_j|\right) .$$

Govil and Mc-tume [3] proved the following generalisations of Theorems B and C:

Theorem D: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$ and $\operatorname{Im}(a_j)=\beta_j$, j=0,1,2,……,n. If for some $k\geq 1$,

$$k\alpha_n \geq \alpha_{n-1}\geq ......\geq \alpha_1\geq \alpha_0,$$

then P(z) has all its zeros in

$$|z+k-1|\leq \frac{k\alpha_n-\alpha_0+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

Theorem E: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$ and $\operatorname{Im}(a_j)=\beta_j$, j=0,1,2,……,n. If for some $k\geq 1$,

$$k\beta_n \geq \beta_{n-1}\geq ......\geq \beta_1\geq \beta_0,$$

then P(z) has all its zeros in

$$|z+k-1|\leq \frac{k\beta_n-\beta_0+|\beta_0|+2\sum_{j=0}^{n}|\alpha_j|}{|\beta_n|}.$$

M. H. Gulzar [4] proved the following generalizations of Theorems D and E:

Theorem F: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$ and $\operatorname{Im}(a_j)=\beta_j$, j=0,1,2,……,n. If for some real numbers $\rho\geq 0$, $0<\mu\leq 1$,

$$\rho+\alpha_n \geq \alpha_{n-1}\geq ......\geq \alpha_1\geq \alpha_0,$$

then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{\alpha_n}\right|\leq \frac{\rho+\alpha_n+2|\alpha_0|-\mu(\alpha_0+|\alpha_0|)+2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

Theorem G: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$ and $\operatorname{Im}(a_j)=\beta_j$, j=0,1,2,……,n. If for some real number $\rho\geq 0$,

$$\rho+\beta_n \geq \beta_{n-1}\geq ......\geq \beta_1\geq \beta_0,$$

then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{\beta_n}\right|\leq \frac{\rho+\beta_n+2|\beta_0|-\mu(\beta_0+|\beta_0|)+2\sum_{j=0}^{n}|\alpha_j|}{|\beta_n|}.$$

In this paper we give generalization of Theorems F and G. In fact, we prove the following:

Theorem 1: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$ and $\operatorname{Im}(a_j)=\beta_j$, j=0,1,2,……,n. If for some real numbers λ, $\rho\geq 0$, $1\leq k\leq n, a_{n-k}\neq 0, 0<\mu\leq 1$,

$$\rho+\alpha_n \geq \alpha_{n-1}\geq ...\alpha_{n-k+1}\geq \lambda\alpha_{n-k}\geq \alpha_{n-k-1}\geq ...\geq \alpha_1\geq \mu\alpha_0,$$

and $\alpha_{n-k-1}>\alpha_{n-k}$, then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right|\leq \frac{\rho+\alpha_n+(\lambda-1)\alpha_{n-k}+|\lambda-1||\alpha_{n-k}|+2|\alpha_0|-\mu(\alpha_0+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|a_n|},$$

and if $\alpha_{n-k}>\alpha_{n-k+1}$, then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right|\leq\frac{\rho+\alpha_n+(1-\lambda)\alpha_{n-k}+|1-\lambda||\alpha_{n-k}|+2|\alpha_0|-\mu(\alpha_0+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|a_n|}$$

Remark 1: Taking $\lambda=1$, Theorem 1 reduces to Theorem F.

If a_j are real i.e. $\beta_j=0$ for all j, we get the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n. If for some real numbers $\lambda,\rho\geq 0$, $1\leq k\leq n, a_{n-k}\neq 0,\ 0<\mu\leq 1,$

$$\rho+a_n\geq a_{n-1}\geq\ldots\geq a_{n-k+1}\geq\lambda a_{n-k}\geq a_{n-k-1}\geq\ldots\geq a_1\geq\mu a_0,$$

and $a_{n-k-1}>a_{n-k}$, then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right|\leq\frac{\rho+a_n+(\lambda-1)a_{n-k}+|\lambda-1||a_{n-k}|+2|a_0|-\mu(a_0+|a_0|)}{|a_n|},,$$

and if $a_{n-k}>a_{n-k+1}$, then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right|\leq\frac{\rho+a_n+(1-\lambda)a_{n-k}+|1-\lambda||a_{n-k}|+2|a_0|-\mu(a_0+|a_0|)}{|a_n|}$$

If we apply Theorem 1 to the polynomial –iP(z) , we easily get the following result:

Theorem 2: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$, j=0,1, 2,……,n. If for some real numbers $\lambda,\rho\geq 0$, $1\leq k\leq n, a_{n-k}\neq 0,\ 0<\mu\leq 1,$

$$\rho+\beta_n\geq\beta_{n-1}\geq\ldots.\geq\beta_{n-k+1}\geq\lambda\beta_{n-k}\geq\beta_{n-k-1}\geq..\geq\beta_1\geq\mu\beta_0,$$

and $\beta_{n-k-1}>\beta_{n-k}$, then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right|\leq\frac{\rho+\beta_n+(\lambda-1)\beta_{n-k}+|\lambda-1||\beta_{n-k}|+2|\beta_0|-\mu(\beta_0+|\beta_0|)+2\sum_{j=0}^{n}|\alpha_j|}{|a_n|},$$

and if $\beta_{n-k}>\beta_{n-k+1}$, then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right|\leq\frac{\rho+\beta_n+(1-\lambda)\beta_{n-k}+|1-\lambda||\beta_{n-k}|+2|\beta_0|-\mu(\beta_0+|\beta_0|)+2\sum_{j=0}^{n}|\alpha_j|}{|a_n|}$$

Remark 2: Taking $\lambda=1$, Theorem 2 reduces to Theorem G.

Theorem 3: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some real numbers $\lambda,\rho\geq 0$, $1\leq k\leq n, a_{n-k}\neq 0,\beta, 0<\mu\leq 1,,$

$$|\rho+a_n|\geq|a_{n-1}|\geq\ldots\geq|a_{n-k+1}|\geq\lambda|a_{n-k}|\geq|a_{n-k-1}|\geq\ldots\geq|a_1|\geq|a_0|$$

and

$$|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\ j=0,1,\ldots\ldots n.$$

If $|a_{n-k-1}|>|a_{n-k}|$ (i.e. $\lambda>1$) , then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right| \le \frac{[|\rho+a_n|(\cos\alpha+\sin\alpha)-|a_{n-k}|(\cos\alpha-\sin\alpha-\lambda\cos\alpha-\lambda\sin\alpha-\lambda+1) + \mu|a_0|(\sin\alpha-\cos\alpha-1)+2|a_0|+2\sin\alpha\sum_{j=1, j\neq n-k}^{n-1}|a_j|]}{|a_n|}$$

If $|a_{n-k}|>|a_{n-k+1}|$ (i.e. $\lambda<1$), then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right| \le \frac{[|\rho+a_n|(\cos\alpha+\sin\alpha)+|a_{n-k}|(\cos\alpha+\sin\alpha-\lambda\cos\alpha+\lambda\sin\alpha+1-\lambda) + \mu|a_0|(\sin\alpha-\cos\alpha-1)+2|a_0|+2\sin\alpha\sum_{j=1, j\neq n-k}^{n-1}|a_j|]}{|a_n|}$$

Remark 3: Taking $\lambda=1$ in Theorem 3 , we get the following result:

Corollary 2: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n. If for some real numbers $\rho\ge 0, 0<\mu\le 1$,

$$|\rho+a_n|\ge|a_{n-1}|\ge.....\ge|a_1|\ge\mu|a_0|,$$

then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right| \le \frac{[|\rho+a_n|(\cos\alpha+\sin\alpha)+\mu|a_0|(\sin\alpha-\cos\alpha-1)+2|a_0|+2\sin\alpha\sum_{j=1,}^{n-1}|a_j|]}{|a_n|}.$$

Remark 4: Taking $\rho=(k-1)a_n, k\ge 1$ in Cor.2, we get the following result:

Corollary 3: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n. If for some real numbers $\rho\ge 0, 0<\mu\le 1$,

$$k|a_n|\ge|a_{n-1}|\ge.....\ge|a_1|\ge\mu|a_0|,$$

then P(z) has all its zeros in the disk

$$\left|z+\frac{\rho}{a_n}\right| \le \frac{[k|a_n|(\cos\alpha+\sin\alpha)+\mu|a_0|(\sin\alpha-\cos\alpha-1)+2|a_0|+2\sin\alpha\sum_{j=1,}^{n-1}|a_j|]}{|a_n|}$$

Taking $\rho=(k-1)a_n, k\ge 1$, and $\mu=1$ in Cor. 3, we get a result of Shah and Liman [7,Theorem 1].

II. LEMMAS

For the proofs of the above results , we need the following results:

Lemma 1: . Let P(z)=$\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with complex coefficients such that

$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}, j=0,1,..n, for$ some real β. Then for some t>0,

$$|ta_j-a_{j-1}|\le\big[t|a_j|-|a_{j-1}|\big]\cos\alpha+\big[t|a_j|+|a_{j-1}|\big]\sin\alpha.$$

The proof of lemma 1 follows from a lemma due to Govil and Rahman [2].

Lemma 2. If p(z) is regular ,p(0)$\neq 0$ and $|p(z)|\le M$ in $|z|\le 1$, then the number of zeros of p(z)in $|z|\le\delta, 0<\delta<1$, does not exceed $\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|p(0)|}$ (see[8],p171).

III. PROOFS OF THEOREMS

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots + a_1 z + a_0) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots\ldots + (a_{n-k+1} - a_{n-k})z^{n-k+1} + (a_{n-k} - a_{n-k-1})z^{n-k} \\ &\quad + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \ldots\ldots + (a_1 - a_0)z + a_0 \\ &= -(\alpha_n + i\beta_n)z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \ldots\ldots + (\alpha_{n-k+1} - \alpha_{n-k})z^{n-k+1} \\ &\quad + (\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} + (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \ldots\ldots + (\alpha_1 - \mu\alpha_0)z \\ &\quad + (\mu - 1)\alpha_0 z + \alpha_0 + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j + i\beta_0 \end{aligned}$$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then $\alpha_{n-k-1} > \alpha_{n-k}$, and we have

$$\begin{aligned} F(z) &= -(\alpha_n + i\beta_n)z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + \ldots\ldots + (\alpha_{n-k+1} - \alpha_{n-k})z^{n-k+1} \\ &\quad + (\lambda\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} - (\lambda - 1)a_{n-k}z^{n-k} + (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \ldots\ldots \\ &\quad + (\alpha_1 - \mu\alpha_0)z + (\mu - 1)\alpha_0 z + \alpha_0 + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j + i\beta_0. \end{aligned}$$

For $|z| > 1$,

$$\begin{aligned} |F(z)| &\geq |a_n z^{n+1} + \rho z^n| - \Big|(\rho + \alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots\ldots + (\alpha_{n-k+1} - \alpha_{n-k})z^{n-k+1} \\ &\quad + (\lambda\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} + (\lambda - 1)a_{n-k}z^{n-k} + (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \ldots\ldots \\ &\quad + (\alpha_1 - \mu\alpha_0)z + (\mu - 1)\alpha_0 z + \alpha_0 + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j + i\beta_0 \Big| \\ &= |z|^n \Bigg[|a_n z + \rho| - \Bigg| (\rho + \alpha_n - \alpha_{n-1}) + (\alpha_{n-1} - \alpha_{n-2})\frac{1}{z} + \ldots\ldots + (\alpha_{n-k+1} - \alpha_{n-k})\frac{1}{z^{k-1}} \\ &\quad + (\lambda\alpha_{n-k} - \alpha_{n-k-1})\frac{1}{z^k} - (\lambda - 1)\alpha_{n-k}\frac{1}{z^k} + (\alpha_{n-k-1} - \alpha_{n-k-2})\frac{1}{k+1} + \ldots. \\ &\quad + (\alpha_1 - \mu\alpha_0)\frac{1}{z^{n-1}} + (\mu - 1)\frac{\alpha_0}{z^{n-1}} + \frac{\alpha_0}{z^n} + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})\frac{1}{z^{n-j}} + i\frac{\beta_0}{z^n} \Bigg| \Bigg] \\ &> |z|^n \Big[|a_n z + \rho| - \{ |\rho + \alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + \ldots\ldots + |\alpha_{n-k+1} - \alpha_{n-k}| \\ &\quad + |\lambda\alpha_{n-k} - \alpha_{n-k-1}| + |\lambda - 1||\alpha_{n-k}| + |\alpha_{n-k-1} - \alpha_{n-k-2}| + \ldots\ldots + |\alpha_1 - \mu\alpha_0| \\ &\quad + |\mu - 1||\alpha_0| + |\alpha_0| + \sum_{j=1}^{n}|\beta_j - \beta_{j-1}| + |\beta_0| \} \Big] \\ &\geq |z|^n \Big[|a_n z + \rho| - \{ \rho + \alpha_n - \alpha_{n-1} + \alpha_{n-1} - \alpha_{n-2} + \ldots\ldots + \alpha_{n-k+1} - \alpha_{n-k} + \lambda\alpha_{n_{n-k}} - \alpha_{n-k-1} \\ &\quad + |\lambda - 1||\alpha_{n-k}| + \alpha_{n-k-1} - \alpha_{n-k-2} + \ldots\ldots + \alpha_1 - \mu\alpha_0 + (1 - \mu)|\alpha_0| \\ &\quad + |\alpha_0| + 2\sum_{j=0}^{n}|\beta_j| \} \Big] \end{aligned}$$

$$= |z|^n \left[|a_n z + \rho| - [\{\rho + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| - \mu(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j| \right]$$

$$>0$$

if

$$|a_n z + \rho| > \rho + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| - \mu(\alpha_0 + |\alpha_0|) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|$$

This shows that the zeros of F(z) whose modulus is greater than 1 lie in

$$\left| z + \frac{\rho}{a_n} \right| \le \frac{\rho + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| + 2|\alpha_0| - \mu(\alpha_0 + |\alpha_0|) + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}.$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence all the zeros of F(z) lie in

$$\left| z + \frac{\rho}{a_n} \right| \le \frac{\rho + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| + 2|\alpha_0| - \mu(\alpha_0 + |\alpha_0|) + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}.$$

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in

$$\left| z + \frac{\rho}{a_n} \right| \le \frac{\rho + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| + 2|\alpha_0| - \mu(\alpha_0 + |\alpha_0|) + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}.$$

If $\alpha_{n-k} > \alpha_{n-k+1}$, then $\alpha_{n-k} > \alpha_{n-k-1}$, and we have

$$\begin{aligned} F(z) = &-(\alpha_n + i\beta_n)z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + \ldots\ldots + (\alpha_{n-k+1} - \lambda\alpha_{n-k})z^{n-k+1} \\ &+ (\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} - (1-\lambda)\alpha_{n-k} z^{n-k+1} + (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \ldots\ldots \\ &+ (\alpha_1 - \mu\alpha_0)z + (\mu - 1)\alpha_0 z + \alpha_0 + i\sum_{j=1}^{n} (\beta_j - \beta_{j-1})z^j + i\beta_0. \end{aligned}$$

For $|z| > 1$,

$$\begin{aligned} |F(z)| \ge &|a_n z^{n+1} + \rho z^n| - |(\rho + \alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots\ldots + (\alpha_{n-k+1} - \lambda\alpha_{n-k})z^{n-k+1} \\ &+ (\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} - (1-\lambda)\alpha_{n-k} z^{n-k+1} + (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \ldots\ldots \\ &+ (\alpha_1 - \mu\alpha_0)z + (\mu - 1)\alpha_0 z + \alpha_0 + i\sum_{j=1}^{n} (\beta_j - \beta_{j-1})z^j + i\beta_0 \ | \\ = &|z|^n \Bigg[|a_n z + \rho| - \Bigg| (\rho + \alpha_n - \alpha_{n-1}) + (\alpha_{n-1} - \alpha_{n-2})\frac{1}{z} + \ldots\ldots + (\alpha_{n-k+1} - \lambda\alpha_{n-k})\frac{1}{z^{k-1}} \\ &+ (\alpha_{n-k} - \alpha_{n-k-1})\frac{1}{z^k} - (1-\lambda)\alpha_{n-k}\frac{1}{z^{k-1}} + (\alpha_{n-k-1} - \alpha_{n-k-2})\frac{1}{z^{k+1}} + \ldots. \\ &+ (\alpha_1 - \mu\alpha_0)\frac{1}{z^{n-1}} + (\mu - 1)\frac{\alpha_0}{z^{n-1}} + \frac{\alpha_0}{z^n} + i\sum_{j=1}^{n} (\beta_j - \beta_{j-1})\frac{1}{z^{n-j}} + i\frac{\beta_0}{z^n} \ \Bigg| \ \Bigg] \\ > &|z|^n \left[|a_n z + \rho| - \{ \ |\rho + \alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + \ldots\ldots + |\alpha_{n-k+1} - \lambda\alpha_{n-k}| \right. \end{aligned}$$

$$+\left|\alpha_{n-k}-\alpha_{n-k-1}\right|+\left|1-\lambda\right|\left|\alpha_{n-k}\right|+\left|\alpha_{n-k-1}-\alpha_{n-k-2}\right|+\ldots\ldots+\left|\alpha_1-\mu\alpha_0\right|$$

$$+\left|\mu-1\right|\left|\alpha_0\right|+\left|\alpha_0\right|+\sum_{j=1}^{n}\left|\beta_j-\beta_{j-1}\right|+\left|\beta_0\right|\ \}\ ]$$

$$\geq|z|^n\Big[\left|a_n z+\rho\right|-\{\rho+\alpha_n-\alpha_{n-1}+\alpha_{n-1}-\alpha_{n-2}+\ldots\ldots+\alpha_{n-k+1}-\lambda\alpha_{n-k}+\alpha_{n_{n-k}}-\alpha_{n-k-1}$$

$$+\left|1-\lambda\right|\left|\alpha_{n-k}\right|+\alpha_{n-k-1}-\alpha_{n-k-2}+\ldots\ldots+\alpha_1-\mu\alpha_0+(1-\mu)\left|\alpha_0\right|$$

$$+\left|\alpha_0\right|+2\sum_{j=0}^{n}\left|\beta_j\right|\ \}\ ]$$

$$=|z|^n\begin{bmatrix}\left|a_n z+\rho\right|-\{\rho+\alpha_n+(1-\lambda)\alpha_{n-k}+\left|1-\lambda\right|\left|\alpha_{n-k}\right|-\mu(\alpha_0+\left|\alpha_0\right|)+ \\ 2\left|\alpha_0\right|+2\sum_{j=0}^{n}\left|\beta_j\right|\}\end{bmatrix}$$

>0

if

$$\left|a_n z+\rho\right|>\rho+\alpha_n+(1-\lambda)\alpha_{n-k}+\left|1-\lambda\right|\left|\alpha_{n-k}\right|-\mu(\alpha_0+\left|\alpha_0\right|)+2\left|\alpha_0\right|+2\sum_{j=0}^{n}\left|\beta_j\right|$$

This shows that the zeros of F(z) whose modulus is greater than 1 lie in

$$\left|z+\frac{\rho}{a_n}\right|\leq\frac{\rho+\alpha_n+(1-\lambda)\alpha_{n-k}+\left|1-\lambda\right|\left|\alpha_{n-k}\right|+2\left|\alpha_0\right|-\mu(\alpha_0+\left|\alpha_0\right|)+2\sum_{j=0}^{n}\left|\beta_j\right|}{\left|a_n\right|}\ .$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence all the zeros of F(z) lie in

$$\left|z+\frac{\rho}{a_n}\right|\leq\frac{\rho+\alpha_n+(1-\lambda)\alpha_{n-k}+\left|1-\lambda\right|\left|\alpha_{n-k}\right|+2\left|\alpha_0\right|-\mu(\alpha_0+\left|\alpha_0\right|)+2\sum_{j=0}^{n}\left|\beta_j\right|}{\left|a_n\right|}\ .$$

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in

$$\left|z+\frac{\rho}{a_n}\right|\leq\frac{\rho+\alpha_n+(1-\lambda)\alpha_{n-k}+\left|1-\lambda\right|\left|\alpha_{n-k}\right|+2\left|\alpha_0\right|-\mu(\alpha_0+\left|\alpha_0\right|)+2\sum_{j=0}^{n}\left|\beta_j\right|}{\left|a_n\right|}\ .$$

That proves Theorem 1.

Proof of Theorem 3: Consider the polynomial

$$F(z)=(1-z)P(z)$$

$$=(1-z)(a_n z^n+a_{n-1}z^{n-1}+\ldots\ldots+a_1 z+a_0)$$

$$=-a_n z^{n+1}+(a_n-a_{n-1})z^n+\ldots\ldots+(a_{n-k+1}-a_{n-k})z^{n-k+1}+(a_{n-k}-a_{n-k-1})z^{n-k}$$

$$+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+\ldots\ldots+(a_1-a_0)z+a_0\ .$$

If $\left|a_{n-k-1}\right|>\left|a_{n-k}\right|$, then $\left|a_{n-k+1}\right|>\left|a_{n-k}\right|$, $\lambda>1$ and we have, for $|z|>1$, by using Lemma1,

$$\left|F(z)\right|\geq\left|a_n z^{n+1}+\rho z^n\right|-\left|(\rho+a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+\ldots\ldots+(a_{n-k+1}-a_{n-k})z^{n-k+1}\right.$$

$$+(\lambda a_{n-k}-a_{n-k-1})z^{n-k}+(\lambda-1)a_{n-k}z^{n-k}+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+\ldots\ldots$$

$$\left.+(a_1-\mu a_0)z+(\mu-1)a_0 z+a_0\ \right|$$

$$= |z|^n \Bigg[|a_n z + \rho| - \Big| (\rho + a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) \frac{1}{z} + \ldots\ldots + (a_{n-k+1} - a_{n-k}) \frac{1}{z^{k-1}}$$

$$+ (\lambda a_{n-k} - a_{n-k-1}) \frac{1}{z^k} + (\lambda - 1) a_{n-k} \frac{1}{z^k} + (a_{n-k-1} - a_{n-k-2}) \frac{1}{z^{k+1}} + \ldots.$$

$$+ (a_1 - \mu a_0) \frac{1}{z^{n-1}} + (\mu - 1) \frac{a_0}{z^{n-1}} + \frac{a_0}{z^n} \Big| \Bigg]$$

$$> |z|^n \Big[|a_n z + \rho| - \{ |\rho + a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \ldots\ldots + |a_{n-k+1} - a_{n-k}|$$

$$+ |\lambda a_{n-k} - a_{n-k-1}| + |\lambda - 1||a_{n-k}| + |a_{n-k-1} - a_{n-k-2}| + \ldots\ldots + |a_1 - \mu a_0|$$

$$+ |\mu - 1||a_0| + |a_0| \} \Big]$$

$$\geq |z|^n \Big[|a_n z + \rho| - \{ (|\rho + a_n| - |a_{n-1}|) \cos\alpha + (|\rho + a_n| + |a_{n-1}|) \sin\alpha$$

$$+ (|a_{n-1}| - |a_{n-2}|) \cos\alpha + (|a_{n-1}| + |a_{n-2}|) \sin\alpha + \ldots\ldots + (|a_{n-k+1}| - |a_{n-k}|) \cos\alpha$$

$$+ (|a_{n-k+1}| + |a_{n-k}|) \sin\alpha + (\lambda |a_{n-k}| - |a_{n-k-1}|) \cos\alpha + (\lambda |a_{n-k}| + |a_{n-k-1}|) \sin\alpha$$

$$+ |\lambda - 1||a_{n-k}| + (|a_{n-k-1}| - |a_{n-k-2}|) \cos\alpha + (|a_{n-k-1}| + |a_{n-k-2}|) \sin\alpha$$

$$+ \ldots\ldots + (|a_1| - \mu |a_0|) \cos\alpha + (|a_1| + \mu |a_0|) \sin\alpha + (1 - \mu)|a_0| + |a_0| \} \Big]$$

$$= |z|^n \Big[|a_n z + \rho| - \{ |\rho + a_n| (\cos\alpha + \sin\alpha) - |a_{n-k}| (\cos\alpha - \sin\alpha - \lambda \cos\alpha - \lambda \sin\alpha$$

$$- \lambda + 1) + \mu |a_0| (\sin\alpha - \cos\alpha + 1) + 2|a_0| + 2 \sin\alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j| \} \Big]$$

$$> 0$$

if

$$|a_n z + \rho| > |\rho + a_n| (\cos\alpha + \sin\alpha) - |a_{n-k}| (\cos\alpha - \sin\alpha - \lambda \cos\alpha - \lambda \sin\alpha$$

$$- \lambda + 1) + \mu |a_0| (\sin\alpha - \cos\alpha - 1) + 2|a_0| + 2 \sin\alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j|$$

This shows that the zeros of F(z) whose modulus is greater than 1 lie in

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{[|\rho + a_n| (\cos\alpha + \sin\alpha) - |a_{n-k}| (\cos\alpha - \sin\alpha - \lambda \cos\alpha - \lambda \sin\alpha - \lambda + 1) + \mu |a_0| (\sin\alpha - \cos\alpha - 1) + 2|a_0| + 2 \sin\alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j|]}{|a_n|} ..$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence all the zeros of F(z) lie in

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{[|\rho + a_n| (\cos\alpha + \sin\alpha) - |a_{n-k}| (\cos\alpha - \sin\alpha - \lambda \cos\alpha - \lambda \sin\alpha - \lambda + 1) + \mu |a_0| (\sin\alpha - \cos\alpha - 1) + 2|a_0| + 2 \sin\alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j|]}{|a_n|} .$$

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of P(z) lie in

$$\left| z + \frac{\rho}{a_n} \right| \leq \frac{[|\rho + a_n| (\cos\alpha + \sin\alpha) - |a_{n-k}| (\cos\alpha - \sin\alpha - \lambda \cos\alpha - \lambda \sin\alpha - \lambda + 1) + \mu |a_0| (\sin\alpha - \cos\alpha - 1) + 2|a_0| + 2 \sin\alpha \sum_{j=1, j \neq n-k}^{n-1} |a_j|]}{|a_n|}$$

If $|a_{n-k}|>|a_{n-k+1}|$, then $|a_{n-k}|>|a_{n-k-1}|$, $\lambda<1$ and we have, for $|z|>1$, by using Lemma 1,

$$\begin{aligned}|F(z)| &\geq |a_n z^{n+1}+\rho z^n| - |(\rho+a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+\ldots\ldots+(a_{n-k+1}-\lambda a_{n-k})z^{n-k+1}\\ &\quad +(a_{n-k}-a_{n-k-1})z^{n-k}-(1-\lambda)a_{n-k}z^{n-k+1}+(a_{n-k-1}-a_{n-k-2})z^{n-k-1}+\ldots\ldots\\ &\quad +(a_1-\mu a_0)z+(\mu-1)a_0 z+a_0\ |\\ &= |z|^n\Big[|a_n z+\rho| - \Big|(\rho+a_n-a_{n-1})+(a_{n-1}-a_{n-2})\frac{1}{z}+\ldots\ldots+(a_{n-k+1}-\lambda a_{n-k})\frac{1}{z^{k-1}}\\ &\quad +(a_{n-k}-a_{n-k-1})\frac{1}{z^k}-(1-\lambda)a_{n-k}\frac{1}{z^{k-1}}+(a_{n-k-1}-a_{n-k-2})\frac{1}{z^{k+1}}+\ldots.\\ &\quad +(a_1-\mu a_0)\frac{1}{z^{n-1}}+(\mu-1)\frac{a_0}{z^{n-1}}+\frac{a_0}{z^n}\ \Big|\ \Big]\\ &> |z|^n\big[|a_n z+\rho| - \{\ |\rho+a_n-a_{n-1}|+|a_{n-1}-a_{n-2}|+\ldots\ldots+|a_{n-k+1}-\lambda a_{n-k}|\\ &\quad +|a_{n-k}-a_{n-k-1}|+|1-\lambda||a_{n-k}|+|a_{n-k-1}-a_{n-k-2}|+\ldots\ldots+|a_1-\mu a_0|\\ &\quad |\mu-1||a_0|+|a_0|\ \}\ \big]\\ &\geq |z|^n\big[|a_n z+\rho| - \{(|\rho+a_n|-|a_{n-1}|)\cos\alpha+(|\rho+a_n|+|a_{n-1}|)\sin\alpha\\ &\quad +(|a_{n-1}|-|a_{n-2}|)\cos\alpha+(|a_{n-1}|+|a_{n-2}|)\sin\alpha+\ldots\ldots+(|a_{n-k+1}|-|\lambda a_{n-k}|)\cos\alpha\\ &\quad +(\ |a_{n-k+1}|+|\lambda a_{n-k}|)\sin\alpha+(|a_{n-k}|-|a_{n-k-1}|)\cos\alpha+(|a_{n-k}|+|a_{n-k-1}|)\sin\alpha\\ &\quad +|1-\lambda||a_{n-k}|+(|a_{n-k-1}|-|a_{n-k-2}|)\cos\alpha+(|a_{n-k-1}|+|a_{n-k-2}|)\sin\alpha\\ &\quad +\ldots\ldots+(|a_1|-\mu|a_0|)\cos\alpha+(|a_1|+\mu|a_0|)\sin\alpha+(1-\mu)|a_0|+|a_0|\}\ \big]\\ &= |z|^n\big[|a_n z+\rho| - \{|\rho+a_n|(\cos\alpha+\sin\alpha)+|a_{n-k}|(\cos\alpha+\sin\alpha-\lambda\cos\alpha+\lambda\sin\alpha\\ &\quad +1-\lambda)+\mu|a_0|(\sin\alpha-\cos\alpha-1)+2|a_0|+2\sin\alpha\sum_{j=1,\,j\neq n-k}^{n-1}|a_j|\}\ \big]\\ &> 0\end{aligned}$$

if

$$|a_n z+\rho| > |\rho+a_n|(\cos\alpha+\sin\alpha)+|a_{n-k}|(\cos\alpha+\sin\alpha-\lambda\cos\alpha+\lambda\sin\alpha+1-\lambda)+\mu|a_0|(\sin\alpha-\cos\alpha-1)+2|a_0|+2\sin\alpha\sum_{j=1,\,j\neq n-k}^{n-1}|a_j|$$

This shows that the zeros of F(z) whose modulus is greater than 1 lie in

$$\left|z+\frac{\rho}{a_n}\right| \leq \frac{[|\rho+a_n|(\cos\alpha+\sin\alpha)+|a_{n-k}|(\cos\alpha+\sin\alpha-\lambda\cos\alpha+\lambda\sin\alpha+1-\lambda)+\mu|a_0|(\sin\alpha-\cos\alpha-1)+2|a_0|+2\sin\alpha\sum_{j=1,\,j\neq n-k}^{n-1}|a_j|]}{|a_n|}.$$

But the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Hence all the zeros of F(z) and therefore P(z) lie in

$$\left|z+\frac{\rho}{a_n}\right| \leq \frac{[|\rho+a_n|(\cos\alpha+\sin\alpha)+|a_{n-k}|(\cos\alpha+\sin\alpha-\lambda\cos\alpha+\lambda\sin\alpha+1-\lambda)+\mu|a_0|(\sin\alpha-\cos\alpha-1)+2|a_0|+2\sin\alpha\sum_{j=1,\,j\neq n-k}^{n-1}|a_j|]}{|a_n|}$$

That proves Theorem 3.

REFERENCES

1) A.Aziz and B.A.Zargar, Some extensions of the Enestrom-Kakeya Theorem , Glasnik Mathematiki, 31(1996) , 239-244.
2) N.K.Govil and Q.I.Rehman, On the Enestrom-Kakeya Theorem, Tohoku Math. J.,20(1968) , 126-136.
3) N.K.Govil and G.N.Mc-tume, Some extensions of the Enestrom- Kakeya Theorem , Int.J.Appl.Math. Vol.11,No.3,2002, 246-253.
4) M. H. Gulzar, On the Zeros of polynomials, International Journal of Modern Engineering Research, Vol.2,Issue 6, Nov-Dec. 2012, 4318- 4322.
5) A. Joyal, G. Labelle, Q.I. Rahman, On the location of zeros of polynomials, Canadian Math. Bull.,10(1967) , 55-63.
6) M. Marden , Geometry of Polynomials, IInd Ed.Math. Surveys, No. 3, Amer. Math. Soc. Providence,R. I,1996.
7) W. M. Shah and A.Liman, On Enestrom-Kakeya Theorem and related Analytic Functions, Proc. Indian Acad. Sci. (Math. Sci.), 117(2007), 359-370.
8) E C. Titchmarsh,The Theory of Functions,2nd edition,Oxford University Press,London,1939.