International Journal of Engineering Inventions
e-ISSN: 2278-7461, p-ISSN: 2319-6491
Volume 2, Issue 5 (March 2013) PP: 54-60

On the Zeros of Analytic Functions inside the Unit Disk

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Abstract: *In this paper we find an upper bound for the number of zeros of an analytic function inside the unit disk by restricting the coefficients of the function to certain conditions.*
AMS Mathematic Subject Classification 2010: *30C 10, 30C15*
Keywords and Phrases*: Bound, Analytic Function, Zeros*

I. Introduction and Statement of Results

A well-known result due to Enestrom and Kakeya [5] states that a polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ of degree n with

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0$$

has all its zeros in $|z| \le 1$.

Q. G. Mohammad [6] initiated the problem of finding an upper bound for the number of zeros of P(z) satisfying the above conditions in $|z| \le \frac{1}{2}$. Many generalizations and refinements were later given by researchers on the bounds for the number of zeros of P(z) in $|z| \le \delta, 0 < \delta < 1$ (for reference see [1]),[2], [4]etc.).

In this paper we consider the same problem for analytic functions and prove the following results:

Theorem 1: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \ne 0$ be analytic for $|z| \le 1$, where $a_j = \alpha_j + i\beta_j$, $j = 0,1,......,n$. If for some $\rho \ge 0$,

$$\rho + \alpha_0 \ge \alpha_1 \ge \alpha_2 \ge ......,$$

then the number of zeros of f(z) in $\frac{|a_0|}{M} \le |z| \le \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{2\rho + |\alpha_0| + \alpha_0 + 2\sum_{j=0}^{\infty} |\beta_j|}{|a_0|},$$

where

$$M = 2\rho + \alpha_0 + |\beta_0| + 2\sum_{j=1}^{\infty} |\beta_j|$$

Taking $\rho = 0$, the following result immediately follows from Theorem 1:

Corollary 1: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \ne 0$ be analytic for $|z| \le 1$, where $a_j = \alpha_j + i\beta_j$, $j = 0,1,......,n$. If

$$\alpha_0 \ge \alpha_1 \ge \alpha_2 \ge ......,$$

then the number of zeros of f(z) in $\frac{|a_0|}{M} \le |z| \le \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|\alpha_0|+\alpha_0+2\sum_{j=0}^{\infty}|\beta_j|}{|a_0|},$$

where

$$M=\alpha_0+|\beta_0|+2\sum_{j=1}^{\infty}|\beta_j| \quad .$$

If the coefficients a_j are real i.e. $\beta_j=0, \forall j=0,1,....,n$, we get the following result from Theorem 1:

Corollary 2 : Let $f(z)=\sum_{j=0}^{\infty}a_j z^j \neq 0$ be analytic for $|z|\leq 1$, where

$$\rho+a_0\geq a_1\geq a_2\geq ......,$$

then the number of zeros of f(z) in $\frac{|a_0|}{M}\leq|z|\leq\delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{2\rho+|a_0|+a_0}{|a_0|},$$

where

$$M=2\rho+a_0 \quad .$$

Taking $\rho=(k-1)\alpha_0$, $k\geq 1$, we get the following result from Theorem 1:

Corollary 3 : Let $f(z)=\sum_{j=0}^{\infty}a_j z^j \neq 0$ be analytic for $|z|\leq 1$, where $a_j=\alpha_j+i\beta_j$, $j=0,1,......,n$. If for some $k\geq 1$,

$$k\alpha_0\geq\alpha_1\geq\alpha_2\geq ......,$$

then the number of zeros of f(z) in $\frac{|a_0|}{M}\leq|z|\leq\delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{(2k-1)\alpha_0+|\alpha_0|+2\sum_{j=0}^{\infty}|\beta_j|}{|a_0|},$$

where

$$M=(2k-1)\alpha_0+|\beta_0|+2\sum_{j=1}^{\infty}|\beta_j|$$

Applying Theorem 1 to the function –if(z), we get the following result:

Theorem 2 : Let $f(z)=\sum_{j=0}^{\infty}a_j z^j \neq 0$ be analytic for $|z|\leq 1$, where $a_j=\alpha_j+i\beta_j$, $j=0,1,......,n$. If for some $\rho\geq 0$,

$$\rho+\beta_0\geq\beta_1\geq\beta_2\geq ......,$$

then the number of zeros of f(z) in $\frac{|a_0|}{M}\leq|z|\leq\delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{2\rho + |\beta_0| + \beta_0 + 2\sum_{j=0}^{\infty} |\alpha_j|}{|a_0|} ,$$

where

$$M = 2\rho + \beta_0 + |\alpha_0| + 2\sum_{j=1}^{\infty} |\alpha_j| \ .$$

Theorem 3 : Let $f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0$ be analytic for $|z| \leq 1$. If for some $\rho \geq 0$,

$$|\rho + a_0| \geq |a_1| \geq |a_2| \geq ...... ,$$

and for real some β,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \ j = 0,1,......., n ,$$

then the number of zeros of f(z) in $\frac{|a_0|}{M} \leq |z| \leq \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{(\rho + |a_0|)(\cos\alpha + \sin\alpha + 1) + \sin\alpha \sum_{j=1}^{\infty} |a_j|}{|a_0|} ,$$

where

$$M = \rho + (\rho + |\alpha_0|)(\cos\alpha + \sin\alpha) + \sin\alpha \sum_{j=1}^{\infty} |a_j| .$$

Taking $\rho = 0$, Theorem 3 reduces to the following result:

Corollary 4: Let $f(z) = \sum_{j=0}^{\infty} a_j z^j \neq 0$ be analytic for $|z| \leq 1$. If,

$$|a_0| \geq |a_1| \geq |a_2| \geq ...... ,$$

and for some real β,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \ j = 0,1,......., n ,$$

then the number of zeros of f(z) in $\frac{|a_0|}{M} \leq |z| \leq \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{|a_0|(\cos\alpha + \sin\alpha + 1) + \sin\alpha \sum_{j=1}^{\infty} |a_j|}{|a_0|} ,$$

where

$$M = |a_0|(\cos\alpha + \sin\alpha) + \sin\alpha \sum_{j=1}^{\infty} |a_j| .$$

Taking $\rho = (k-1)|a_0|$, $k \geq 1$, we get the following result from Theorem 3:

Corollary 5: Let $f(z)=\sum_{j=0}^{\infty}a_j z^j \neq 0$ be analytic for $|z|\leq 1$. If,

$$k|a_0|\geq|a_1|\geq|a_2|\geq......,$$

and for some real β,

$$|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\ j=0,1,.......,n,$$

then the number of zeros of f(z) in $\frac{|a_0|}{M}\leq|z|\leq\delta, 0<\delta<1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{k|a_0|(\cos\alpha+\sin\alpha+1)+\sin\alpha\sum_{j=1}^{\infty}|a_j|}{|a_0|},$$

where

$$M=2k|\alpha_0|(\cos\alpha+\sin\alpha)-|a_0|+\sin\alpha\sum_{j=1}^{\infty}|a_j|.$$

2. Lemmas

For the proofs of the above results we need the following results:

Lemma 1 : Let f(z) be analytic for $|z|\leq 1$, $f(0)\neq 0$ and $|f(z)|\leq M$ for $|z|\leq 1$,

Then the number of zeros of f(z) in $|z|\leq\delta, 0<\delta<1$, does not exceed

$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|f(0)|}$ (see [7] , page 171).

Lemma 2 : If for some t>0, $|ta_j|\geq|a_{j-1}|$ and $|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2}, j=0,1,2,...$, for some real β, then

$$|ta_j-a_{j-1}|\leq(|ta_j|-|a_{j-1}|)\cos\alpha+(|ta_j|+|a_{j-1}|)\sin\alpha.$$

The proof of Lemma 2 follows from a lemma of Govil and Rahman [3].

3. Proofs of Theorems:

Proof of Theorem 1: Consider the function

$$\begin{aligned}F(z)&=(1-z)f(z)\\&=(1-z)(a_0+a_1z+a_2z^2+......)\\&=a_0-(a_0-a_1)z-(a_1-a_2)z^2+......\\&=\alpha_0+\rho z-(\rho+\alpha_0-\alpha_1)z-(\alpha_1-\alpha_2)z^2-......\\&\quad+i\beta_0-i\{(\beta_0-\beta_1)z+(\beta_1-\beta_2)z+......\}.\end{aligned}$$

For $|z|\leq 1$,

$$\begin{aligned}|F(z)|&\leq\rho+|\alpha_0|+\rho+\alpha_0-\alpha_1+\alpha_1-\alpha_2+\alpha_2-\alpha_3+......+|\beta_0|\\&\quad+|\beta_0|+|\beta_1|+|\beta_1|+|\beta_2|+......\\&=2\rho+|\alpha_0|+\alpha_0+2\sum_{j=0}^{\infty}|\beta_j|.\end{aligned}$$

Since F(z) is analytic for $|z| \le 1$, $F(0) = a_0 \ne 0$, it follows , by using Lemma 1, that the number of zeros of F(z) in $|z| \le \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{2\rho + |\alpha_0| + \alpha_0 + 2\sum_{j=0}^{\infty} |\beta_j|}{|a_0|}.$$

On the other hand, consider

$$\begin{aligned} F(z) &= (1-z) f(z) \\ &= (1-z)(a_0 + a_1 z + a_2 z^2 + ...) \\ &= a_0 - (a_0 - a_1) z - (a_1 - a_2) z^2 + ... \\ &= a_0 + q(z), \end{aligned}$$

where

$$\begin{aligned} q(z) &= -(a_0 - a_1) z - (a_1 - a_2) z^2 + ... \\ &= \rho z - (\rho + \alpha_0 - \alpha_1) z - (\alpha_1 - \alpha_2) z^2 - ...... \\ &\quad - i\{(\beta_0 - \beta_1) z + (\beta_1 - \beta_2) z^2 + ...\} \end{aligned}$$

For $|z| = 1$,

$$\begin{aligned} |q(z)| &\le \rho + \rho + \alpha_0 - \alpha_1 + \alpha_1 - \alpha_2 + ... + |\beta_0| + |\beta_1| + |\beta_1| + |\beta_2| + ... \\ &= 2\rho + \alpha_0 + |\beta_0| + 2\sum_{j=1}^{\infty} |\beta_j| = M. \end{aligned}$$

Since q(z) is analytic for $|z| \le 1$, q(0)=0, it follows , by Schwarz's lemma , that

$$|q(z) \le M|z|| \text{ for } |z| \le 1.$$

Hence for $|z| \le 1$,

$$\begin{aligned} |F(z)| &= |a_0 + q(z)| \\ &\ge |a_0| - |q(z)| \\ &\ge |a_0| - M|z| \\ &> 0 \end{aligned}$$

if

$$|z| < \frac{|a_0|}{M}.$$

This shows that all the zeros of F(z) lie in $|z| \ge \frac{|a_0|}{M}$. Since the zeros of f(z) are also the zeros of F(z), it follows that all the zeros of f(z) lie in $|z| \ge \frac{|a_0|}{M}$. Thus, the number of zeros of f(z) in $\frac{|a_0|}{M} \le |z| \le \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{2\rho+|\alpha_0|+\alpha_0+2\sum_{j=0}^{\infty}|\beta_j|}{|a_0|}.$$

Proof of Theorem 2: Consider the function

$$\begin{aligned}F(z)&=(1-z)f(z)\\&=(1-z)(a_0+a_1z+a_2z^2+......)\\&=a_0-(a_0-a_1)z-(a_1-a_2)z^2+......\\&=a_0+\rho z-(\rho+a_0-a_1)z-(a_1-a_2)z^2+......\end{aligned}$$

For $|z|\le 1$, we have, by using the hypothesis and Lemma 2,

$$\begin{aligned}|F(z)|&\le \rho+|a_0|+[(|\rho+a_0|-|a_1|)\cos\alpha+(|\rho+a_0|+|a_1|)\sin\alpha\\&\quad+(|a_1|-|a_2|)\cos\alpha+(|a_1|+|a_2|)\sin\alpha+......\\&\le(\rho+|a_0|)(\cos\alpha+\sin\alpha+1)+\sin\alpha\sum_{j=1}^{\infty}|a_j|.\end{aligned}$$

Since F(z) is analytic for $|z|\le 1$, $F(0)=a_0\neq 0$, it follows , by using the lemma 1, that the number of zeros of F(z) in $|z|\le\delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{(\rho+|a_0|)(\cos\alpha+\sin\alpha+1)+\sin\alpha\sum_{j=1}^{\infty}|a_j|}{|a_0|}.$$

Again , Consider the function

$$\begin{aligned}F(z)&=(1-z)f(z)\\&=(1-z)(a_0+a_1z+a_2z^2+......)\\&=a_0-(a_0-a_1)z-(a_1-a_2)z^2+......\\&=a_0+\rho z-(\rho+a_0-a_1)z-(a_1-a_2)z^2+......\\&=a_0+q(z),\end{aligned}$$

where

$$\begin{aligned}q(z)&=-(a_0-a_1)z-(a_1-a_2)z^2+......\\&=\rho z-(\rho+a_0-a_1)z-(a_1-a_2)z^2-......\end{aligned}$$

For $|z|=1$, by using lemma 2, we have,

$$\begin{aligned}|q(z)|&\le\rho+(|\rho+\alpha_0|-|a_1|)\cos\alpha+(|\rho+\alpha_0|+|a_1|)\sin\alpha\\&\quad+(|a_1|-|a_2|)\cos\alpha+(|a_1|+|a_2|)\sin\alpha+......\\&\le\rho+(\rho+|a_0|)(\cos\alpha+\sin\alpha)+\sin\alpha\sum_{j=1}^{\infty}|a_j|=M.\end{aligned}$$

Since q(z) is analytic for $|z|\le 1$, q(0)=0, it follows , by Schwarz's lemma , that

$|q(z)\le M|z||$ for $|z|\le 1$.

Hence for $|z|\le 1$,

$$|F(z)|=|a_0+q(z)|$$

$$\geq |a_0| - |q(z)|$$

$$\geq |a_0| - M|z|$$

$$> 0$$

if

$$|z| < \frac{|a_0|}{M}.$$

This shows that all the zeros of F(z) lie in $|z| \geq \frac{|a_0|}{M}$. Since the zeros of f(z) are also the zeros of F(z), it follows that all the zeros of f(z) lie in $|z| \geq \frac{|a_0|}{M}$. Thus, the number of zeros of f(z) in $\frac{|a_0|}{M} \leq |z| \leq \delta, 0 < \delta < 1$, does not exceed

$$\frac{1}{\log \frac{1}{\delta}} \log \frac{(\rho + |a_0|)(\cos\alpha + \sin\alpha + 1) + \sin\alpha \sum_{j=1}^{\infty} |a_j|}{|a_0|}.$$

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