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Zeros of a Polynomial in a given Circle

M. H. Gulzar
Department of Mathematics, University of Kashmir, Srinagar 190006

Abstract: *In this paper we discuss the problem of finding the number of zeros of a polynomial in a given circle when the coefficients of the polynomial or their real or imaginary parts are restricted to certain conditions. Our results in this direction generalize some well- known results in the theory of the distribution of zeros of polynomials.*
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I. Introduction and Statement of Results

In the literature many results have been proved on the number of zeros of a polynomial in a given circle. In this direction Q. G. Mohammad [5] has proved the following result:

Theorem A: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0,$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{a_n}{a_0}.$$

K. K. Dewan [2] generalized Theorem A to polynomials with complex coefficients and proved the following results:

Theorem B: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_1 \ge \alpha_0 > 0,$$

Then the number of zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$1+\frac{1}{\log 2}\log\frac{\alpha_n+\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

Theorem C: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α, β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j=0,1,2,......,n$$

and

$$|a_n| \ge |a_{n-1}| \ge ...... \ge |a_1| \ge |a_0| .$$

Then the number f zeros of P(z) in $|z| \le \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\log\frac{|a_n|(\cos\alpha+\sin\alpha+1)+2\sin\alpha\sum_{j=0}^{n-1}|a_j|}{|a_0|} .$$

The above results were further generalized by researchers in various ways. M. H. Gulzar[4] proved the following results:

Theorem D: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\operatorname{Re}(a_j)=\alpha_j$, $\operatorname{Im}(a_j)=\beta_j$ and

$$\rho+\alpha_n \ge \alpha_{n-1} \ge ......\alpha_{n-k+1} \ge \lambda\alpha_{n-k} \ge \alpha_{n-k-1}...... \ge \alpha_1 \ge \tau\alpha_0 ,$$

for some real numbers $\lambda, \rho \ge 0, 0<\tau \le 1, 1 \le k \le n, \alpha_{n-k} \ne 0.$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then the number f zeros of P(z) in $|z| \le \delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}} \log \frac{2\rho+|\alpha_n|+\alpha_n+(\lambda-1)\alpha_{n-k}+|\lambda-1||\alpha_{n-k}|-\tau(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}$$

and if $\alpha_{n-k} > \alpha_{n-k+1}$, then the number f zeros of P(z) in $|z| \le \delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}} \log \frac{2\rho+|\alpha_n|+\alpha_n+(1-\lambda)\alpha_{n-k}+|1-\lambda||\alpha_{n-k}|-\tau(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}$$

Theorem E: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α, β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j=0,1,2,......,n$$

and

$$|\rho+a_n| \ge |a_{n-1}| \ge ...... \ge |a_{n-k+1}| \ge \lambda|a_{n-k}| \ge |a_{n-k-1}| \ge ......|a_1| \ge \tau|a_0| ,$$

for some $\rho \ge 0, \lambda>0, 1 \le k \le n, \alpha_{n-k} \ne 0, 0<\tau \le 1$.

If $|a_{n-k-1}|>|a_{n-k}| i.e. \lambda>1$, then the number of zeros of P(z) in $|z| \le \delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}} \log \frac{M_1}{|a_0|} ,$$

where

$$M_1=(\rho+|a_n|)(\cos\alpha+\sin\alpha+1)-|a_{n-k}|(\cos\alpha-\sin\alpha-\lambda\cos\alpha-\lambda\sin\alpha-\lambda+1)$$
$$-\tau|a_0|(\cos\alpha-\sin\alpha+1)+2|a_0|+2\sin\alpha\sum_{j=1, j\ne n-k}^{n-1}|a_j| ,$$

and

if $|a_{n-k}|>|a_{n-k+1}| i.e. \lambda<1$, then the number of zeros of P(z) in $|z| \le \delta, 0<\delta<1$, does not exceed

$$\frac{1}{\log\frac{1}{\delta}} \log \frac{M_2}{|a_0|} ,$$

where

$$M_2=(\rho+|a_n|)(\cos\alpha+\sin\alpha+1)+|a_{n-k}|(\cos\alpha+\sin\alpha-\lambda\cos\alpha+\lambda\sin\alpha-\lambda+1)$$
$$-\tau|a_0|(\cos\alpha-\sin\alpha+1)+2|a_0|+2\sin\alpha\sum_{j=1, j\ne n-k}^{n-1}|a_j| .$$

In the present paper , we find bounds for the number of zeros of P(z) of the above results in a circle of any positive radius. More precisely, we prove the following resuls:

Theorem 1: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\text{Re}(a_j)=\alpha_j$, $\text{Im}(a_j)=\beta_j$ and

$$\rho+\alpha_n \geq \alpha_{n-1} \geq ......\alpha_{n-k+1} \geq \lambda\alpha_{n-k} \geq \alpha_{n-k-1}...... \geq \alpha_1 \geq \tau\alpha_0 ,$$

for some real numbers $\lambda, \rho \geq 0, 0<\tau \leq 1, 1 \leq k \leq n, \alpha_{n-k} \neq 0..$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then the number f zeros of P(z) in $|z| \leq \frac{R}{c} (R>0, c>1)$, does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[2\rho+|\alpha_n|+\alpha_n+(\lambda-1)\alpha_{n-k}+|\lambda-1||\alpha_{n-k}|-\tau(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \geq 1$

and

$$\frac{1}{\log c}\log\frac{|a_0|+R[2\rho+|\alpha_n|+\alpha_n+(\lambda-1)\alpha_{n-k}+|\lambda-1||\alpha_{n-k}|-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \leq 1.$

If $\alpha_{n-k} > \alpha_{n-k+1}$, then the number f zeros of P(z) in $|z| \leq \frac{R}{c} (R>0, c>1)$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[2\rho+|\alpha_n|+\alpha_n+(1-\lambda)\alpha_{n-k}+|1-\lambda||\alpha_{n-k}|-\tau(|\alpha_0|+\alpha_0)+2|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \geq 1$

and

$$\frac{1}{\log c}\log\frac{|a_0|+R[2\rho+|\alpha_n|+\alpha_n+(1-\lambda)\alpha_{n-k}+|1-\lambda||\alpha_{n-k}|-\tau(|\alpha_0|+\alpha_0)+|\alpha_0|+|\beta_0|+2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \leq 1.$

Remark 1: Taking R=1and $c=\frac{1}{\delta}, 0<\delta<1$, Theorem 1 reduces to Theorem D.

If the coefficients a_j are real i.e. $\beta_j = 0, \forall j$, then we get the following result from Theorem 1:

Corollary 1: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n such that $\text{Re}(a_j)=\alpha_j$, $\text{Im}(a_j)=\beta_j$ and

$$\rho+a_n \geq a_{n-1} \geq ......a_{n-k+1} \geq \lambda a_{n-k} \geq a_{n-k-1}...... \geq a_1 \geq \tau a_0 ,$$

for some real numbers $\lambda, \rho \geq 0, 0<\tau \leq 1, 1 \leq k \leq n, a_{n-k} \neq 0..$

If $a_{n-k-1} > a_{n-k}$, then the number f zeros of P(z) in $|z| \leq \frac{R}{c} (R>0, c>1)$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[2\rho+|a_n|+a_n+(\lambda-1)a_{n-k}+|\lambda-1||a_{n-k}|-\tau(|a_0|+a_0)+2|a_0|}{|a_0|}$$

for $R \geq 1$

and

$$\frac{1}{\log c}\log\frac{|a_0|+R[2\rho+|a_n|+a_n+(\lambda-1)a_{n-k}+|\lambda-1||a_{n-k}|-\tau(|a_0|+a_0)+|a_0|}{|a_0|}$$

for $R\le 1$.

If $a_{n-k}>a_{n-k+1}$, then the number f zeros of P(z) in $|z|\le\frac{R}{c}(R>0,c>1)$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[2\rho+|a_n|+a_n+(1-\lambda)a_{n-k}+|1-\lambda||a_{n-k}|-\tau(|a_0|+a_0)+2|a_0|}{|a_0|}$$

for $R\ge 1$

and

$$\frac{1}{\log c}\log\frac{|a_0|+R[2\rho+|a_n|+a_n+(1-\lambda)a_{n-k}+|1-\lambda||a_{n-k}|-\tau(|a_0|+a_0)+|a_0|}{|a_0|}$$

for $R\le 1$.

Applying Theorem 1 to the polynomial –iP(z) , we get the following result:

Theorem 2: : Let $P(z)=\sum_{j=0}^{\infty}a_jz^j$ be a polynomial o f degree n such that $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$ and

$$\rho+\beta_n\ge\beta_{n-1}\ge......\beta_{n-k+1}\ge\lambda\beta_{n-k}\ge\beta_{n-k-1}......\ge\beta_1\ge\tau\beta_0,$$

for some real numbers $\lambda,\rho\ge 0, 0<\tau\le 1, 1\le k\le n, \beta_{n-k}\ne 0.$.

If $\beta_{n-k-1}>\beta_{n-k}$, then the number f zeros of P(z) in $|z|\le\frac{R}{c}(R>0,c>1)$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[2\rho+|\beta_n|+\beta_n+(\lambda-1)\beta_{n-k}+|\lambda-1||\beta_{n-k}|-\tau(|\beta_0|+\beta_0)+2|\beta_0|+2\sum_{j=0}^{n}|\alpha_j|]}{|a_0|}$$

for $R\ge 1$

and

$$\frac{1}{\log c}\log\frac{|a_0|+R[2\rho+|\beta_n|+\beta_n+(\lambda-1)\beta_{n-k}+|\lambda-1||\beta_{n-k}|-\tau(|\beta_0|+\beta_0)+|\beta_0|+|\alpha_0|+2\sum_{j=0}^{n}|\alpha_j|]}{|a_0|}$$

for $R\le 1$.

If $\beta_{n-k}>\beta_{n-k+1}$, then the number f zeros of P(z) in $|z|\le\frac{R}{c}(R>0,c>1)$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[2\rho+|\beta_n|+\beta_n+(1-\lambda)\beta_{n-k}+|1-\lambda||\beta_{n-k}|-\tau(|\beta_0|+\beta_0)+2|\beta_0|+2\sum_{j=0}^{n}|\alpha_j|]}{|a_0|}$$

for $R\ge 1$

and

$$\frac{1}{\log c}\log\frac{|a_0|+R[2\rho+|\beta_n|+\beta_n+(1-\lambda)\beta_{n-k}+|1-\lambda||\beta_{n-k}|-\tau(|\beta_0|+\beta_0)+|\beta_0|+|\alpha_0|+2\sum_{j=0}^{n}|\alpha_j|]}{|a_0|}$$

for $R\le 1$.

Theorem 3: Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α, β,

$$\left|\arg a_j - \beta\right| \le \alpha \le \frac{\pi}{2},\ j=0,1,2,......,n$$

and

$$\left|\rho + a_n\right| \ge \left|a_{n-1}\right| \ge ...... \ge \left|a_{n-k+1}\right| \ge \lambda\left|a_{n-k}\right| \ge \left|a_{n-k-1}\right| \ge .....\left|a_1\right| \ge \tau\left|a_0\right| ,$$

for some $\rho \ge 0, \lambda > 0, 1 \le k \le n, \alpha_{n-k} \ne 0, 0 < \tau \le 1$.

If $\left|a_{n-k-1}\right| > \left|a_{n-k}\right| i.e. \lambda > 1$, then the number of zeros of P(z) in $|z| \le \frac{R}{c} (R>0, c>1)$, does not exceed

$$\frac{1}{\log c} \log \frac{M_3}{|a_0|} ,$$

where

$$M_3 = R^{n+1}[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1)$$
$$-\tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1} |a_j|], \text{ for } R \ge 1$$

and

$$M_3 = |a_0| + R[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1)$$
$$-\tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1} |a_j|], \text{ for } R \le 1.$$

If $\left|a_{n-k}\right| > \left|a_{n-k+1}\right| i.e. \lambda < 1$, then the number of zeros of P(z) in $|z| \le \frac{R}{c} (R>0, c>1)$ does not exceed

$$\frac{1}{\log c} \log \frac{M_4}{|a_0|} ,$$

Where

$$M_4 = R^{n+1}[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha - \lambda + 1)$$
$$-\tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1} |a_j|] \quad \text{for } R \ge 1$$

and

$$M_4 = |a_0| + R[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha - \lambda + 1)$$
$$-\tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1} |a_j|] \quad \text{for } R \le 1.$$

Remark 2: Taking R=1and $c=\frac{1}{\delta}, 0<\delta<1$, Theorem 1 reduces to Theorem E.

For different values of the parameters R, c, k, λ, τ, we get many other interesting results.

2. Lemmas

For the proofs of the above results we need the following results:

Lemma 1: If f(z) is analytic in $|z| \le R$,but not identically zero, f(0) $\ne$ 0 and $f(a_k)=0, k=1,2,......,n$, then

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(\mathrm{Re}^{i\theta}\right| d\theta - \log\left|f(0)\right| = \sum_{j=1}^{n} \log\frac{R}{|a_j|}.$$

Lemma 1 is the famous Jensen's theorem (see page 208 of [1]).

Lemma 2: If f(z) is analytic and $|f(z)| \le M(r)$ in $|z| \le r$, then the number of zeros of F(z) in $|z| \le \frac{r}{c}, c > 1$ does not exceed

$$\frac{1}{\log c} \log \frac{M(r)}{|f(0)|}.$$

Lemma 2 is a simple deduction from Lemma 1.

Lemma 3: Let $P(z) = \sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α, β, $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, 0 \le j \le n,$ and

$|a_j| \ge |a_{j-1}|, 0 \le j \le n,$ then any t>0,

$$|ta_j - a_{j-1}| \le (t|a_j| - |a_{j-1}|)\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha .\text{s}$$

Lemma 3 is due to Govil and Rahman [3].

3.Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1} z^{n-1} + \dots\dots + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \dots\dots + (a_{n-k+1} - a_{n-k})z^{n-k+1} + (a_{n-k} - a_{n-k-1})z^{n-k} \\
&\quad + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \dots\dots + (a_1 - a_0)z + a_0 \\
&= -(\alpha_n + i\beta_n)z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \dots\dots + (\alpha_{n-k+1} - \alpha_{n-k})z^{n-k+1} \\
&\quad + (\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} + (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \dots\dots + (\alpha_1 - \tau\alpha_0)z \\
&\quad + (\tau - 1)\alpha_0 z + \alpha_0 + i\sum_{j=1}^{n} (\beta_j - \beta_{j-1})z^j + i\beta_0
\end{aligned}$$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then

$$\begin{aligned}
F(z) &= -(\alpha_n + i\beta_n)z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + \dots\dots + (\alpha_{n-k+1} - \alpha_{n-k})z^{n-k+1} \\
&\quad + (\lambda\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} - (\lambda - 1)\alpha_{n-k} z^{n-k} + (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \dots\dots \\
&\quad + (\alpha_1 - \tau\alpha_0)z + (\tau - 1)\alpha_0 z + \alpha_0 + i\sum_{j=1}^{n} (\beta_j - \beta_{j-1})z^j + i\beta_0.
\end{aligned}$$

For $|z| \le R$, we have by using the hypothesis

$$\begin{aligned}
|F(z)| &\le |\alpha_n| R^{n+1} + |\beta_n| R^{n+1} + |\rho| R^n + |\rho + \alpha_n - \alpha_{n-1}| R^n + \dots\dots + |\alpha_{n-k+1} - \alpha_{n-k}| R^{n-k+1} \\
&\quad + |\lambda\alpha_{n-k} - \alpha_{n-k-1}| R^{n-k} + |\lambda - 1||\alpha_{n-k}| R^{n-k} \\
&\quad + |\alpha_{n-k-1} - \alpha_{n-k-2}| R^{n-k-1} + \dots\dots + |\alpha_1 - \tau\alpha_0| R + (1-\tau)|\alpha_0| R \\
&\quad + |\alpha_0| + |\beta_0| + \sum_{j=0}^{n} (|\beta_j| + |\beta_{j-1}|) R^j \\
&\le R^{n+1}[2\rho + |\alpha_n| + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|]
\end{aligned}$$

for $R \ge 1$

and

$$\leq |a_0| + R[2\rho + |\alpha_n| + \alpha_n + (\lambda-1)\alpha_{n-k} + |\lambda-1||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n}|\beta_j|$$

for $R \leq 1$.

Hence by Lemma 2, the number of zeros of F(z) in $|z| \leq \dfrac{R}{c}(R>0, c>1)$, does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[2\rho + |\alpha_n| + \alpha_n + (\lambda-1)\alpha_{n-k} + |\lambda-1||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \geq 1$

and

$$\frac{1}{\log c}\log\frac{|a_0| + R[2\rho + |\alpha_n| + \alpha_n + (\lambda-1)\alpha_{n-k} + |\lambda-1||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \leq 1$.

If $\alpha_{n-k} > \alpha_{n-k+1}$, then

$$F(z) = -(\alpha_n + i\beta_n)z^{n+1} - \rho z^n + (\rho + \alpha_n - \alpha_{n-1})z^n + \ldots\ldots + (\alpha_{n-k+1} - \lambda\alpha_{n-k})z^{n-k+1}$$
$$+ (\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} - (1-\lambda)\alpha_{n-k}z^{n-k+1} + (\alpha_{n-k-1} - \alpha_{n-k-2})z^{n-k-1} + \ldots\ldots$$
$$+ (\alpha_1 - \alpha_0)z + \alpha_0 + i\sum_{j=1}^{n}(\beta_j - \beta_{j-1})z^j + i\beta_0.$$

For $|z| \leq R$, we have by using the hypothesis

$$|F(z)| \leq |\alpha_n|R^{n+1} + |\beta_n|R^{n+1} + |\rho|R^n + |\rho + \alpha_n - \alpha_{n-1}|R^n + \ldots\ldots + |\alpha_{n-k+1} - \lambda\alpha_{n-k}|R^{n-k+1}$$
$$+ |\alpha_{n-k} - \alpha_{n-k-1}|R^{n-k} + |1-\lambda||\alpha_{n-k}|R^{n-k+}$$
$$+ |\alpha_{n-k-1} - \alpha_{n-k-2}|R^{n-k-1} + \ldots\ldots + |\alpha_1 - \tau\alpha_0|R + (1-\tau)|\alpha_0|R$$
$$+ |\alpha_0| + |\beta_0| + \sum_{j=0}^{n}(|\beta_j| + |\beta_{j-1}|)R^j$$
$$\leq R^{n+1}[2\rho + |\alpha_n| + \alpha_n + (1-\lambda)\alpha_{n-k} + |1-\lambda||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|]$$

for $R \geq 1$

and

$$\leq |a_0| + R[2\rho + |\alpha_n| + \alpha_n + (1-\lambda)\alpha_{n-k} + |1-\lambda||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=1}^{n}|\beta_j|$$

for $R \leq 1$.

Hence by Lemma 2, the number of zeros of F(z) in $|z| \leq \dfrac{R}{c}(R>0, c>1)$ does not exceed

$$\frac{1}{\log c}\log\frac{R^{n+1}[2\rho + |\alpha_n| + \alpha_n + (1-\lambda)\alpha_{n-k} + |1-\lambda||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \geq 1$

and

$$\frac{1}{\log c}\log\frac{|a_0| + R[2\rho + |\alpha_n| + \alpha_n + (1-\lambda)\alpha_{n-k} + |1-\lambda||\alpha_{n-k}| - \tau(|\alpha_0| + \alpha_0) + |\alpha_0| + |\beta_0| + 2\sum_{j=0}^{n}|\beta_j|]}{|a_0|}$$

for $R \leq 1$.

That proves Theorem 1 completely.

Proof of Theorem 4: Consider the polynomial

$$F(z) = (1-z)P(z)$$
$$= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0)$$
$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{n-k+1} - a_{n-k})z^{n-k+1} + (a_{n-k} - a_{n-k-1})z^{n-k}$$
$$+ (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + ... + (a_1 - a_0)z + a_0$$

If $|a_{n-k-1}| > |a_{n-k}|$, i.e. $\lambda > 1$, then

$$F(z) = -a_n z^{n+1} - \rho z^n + (\rho + a_n - a_{n-1})z^n + ... + (a_{n-k+1} - a_{n-k})z^{n-k+1}$$
$$+ (\lambda a_{n-k} - a_{n-k-1})z^{n-k} - (\lambda - 1)a_{n-k}z^{n-k} + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + ...$$
$$+ (a_1 - \tau a_0)z + (\tau - 1)a_0 z + a_0$$

so that for $|z| \le R$, we have by using Lemma 3,

$$|F(z)| \le |a_n|R^{n+1} + \rho R^n + |\rho + a_n - a_{n-1}|R^n + ... + |a_{n-k+1} - a_{n-k}|R^{n-k+1} + |\lambda a_{n-k} - a_{n-k-1}|R^{n-k}$$
$$+ |\lambda - 1||a_{n-k}|R^{n-k} + |a_{n-k-1} - a_{n-k-2}|R^{n-k-1} + ... + |a_1 - \tau a_0|R + (1-\tau)|a_0|R + |a_0|$$
$$\le R^{n+1}[|a_n| + \rho + (|\rho + a_n| - |a_{n-1}|)\cos\alpha + (|\rho + a_n| + |a_{n-1}|)\sin\alpha + ...$$
$$+ (|a_{n-k+1}| - |a_{n-k}|)\cos\alpha + (|a_{n-k+1}| + |a_{n-k}|)\sin\alpha + (\lambda - 1)|a_{n-k}|$$
$$+ (\lambda|a_{n-k}| - |a_{n-k-1}|)\cos\alpha + (\lambda|a_{n-k}| + |a_{n-k-1}|)\sin\alpha$$
$$+ (|a_{n-k-1}| - |a_{n-k-2}|)\cos\alpha + (|a_{n-k-1}| + |a_{n-k-2}|)\sin\alpha + ...$$
$$+ (|a_1| - \tau|a_0|)\cos\alpha + (|a_1| + \tau|a_0|)\sin\alpha + (1-\tau)|a_0| + |a_0|]$$
$$\le R^{n+1}[(\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha$$
$$- \lambda\sin\alpha - \lambda + 1) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1}|a_j|\,]$$

for $R \ge 1$

and

$$\le |a_0| + R[(\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha$$
$$- \lambda\sin\alpha - \lambda + 1) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1}|a_j|\,]$$

for $R \le 1$

Hence , by Lemma 2, the number of zeros of F(z) and hence P(z) in $|z| \le \frac{R}{c}\,(R > 0, c > 1)$ does not exceed

$$\frac{1}{\log c}\log\frac{M_3}{|a_0|} \quad ,$$

where

$$M_3 = R^{n+1}[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1)$$
$$-\tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1}|a_j|], \quad \text{for } R \ge 1$$

and

$$M_3 = |a_0| + R[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) - |a_{n-k}|(\cos\alpha - \sin\alpha - \lambda\cos\alpha - \lambda\sin\alpha - \lambda + 1)$$
$$-\tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1}|a_j|], \quad \text{for } R \le 1.$$

If $|a_{n-k}| > |a_{n-k+1}|$, i.e. $\lambda < 1$, then

$$F(z) = -a_n z^{n+1} - \rho z^n + (\rho + a_n - a_{n-1})z^n + ... + (a_{n-k+1} - \lambda a_{n-k})z^{n-k+1}$$
$$+ (a_{n-k} - a_{n-k-1})z^{n-k} - (1-\lambda)a_{n-k} z^{n-k+1} + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + ...$$
$$+ (a_1 - a_0)z + a_0$$

so that for $|z| \le R$, we have by hypothesis and Lemma 3,

$$|F(z)| \le |a_n| R^{n+1} + \rho R^n + |\rho + a_n - a_{n-1}| R^n + ... + |a_{n-k+1} - \lambda a_{n-k}| R^{n-k+1} + |a_{n-k} - a_{n-k-1}| R^{n-k}$$
$$+ |1-\lambda||a_{n-k}| R^{n-k+1} + |a_{n-k-1} - a_{n-k-2}| R^{n-k-1} + ...$$
$$+ |a_1 - \tau a_0| R + (1-\tau)|a_0| R + |a_0|$$
$$\le R^{n+1}[|a_n| + \rho + (|\rho + a_n| - |a_{n-1}|)\cos\alpha + (|\rho + a_n| + |a_{n-1}|)\sin\alpha + ...$$
$$+ (|a_{n-k+1}| - \lambda|a_{n-k}|)\cos\alpha + (|a_{n-k+1}| + \lambda|a_{n-k}|)\sin\alpha + |1-\lambda||a_{n-k}|$$
$$+ (|a_{n-k}| - |a_{n-k-1}|)\cos\alpha + (|a_{n-k}| + |a_{n-k-1}|)\sin\alpha$$
$$+ (|a_{n-k-1}| - |a_{n-k-2}|)\cos\alpha + (|a_{n-k-1}| + |a_{n-k-2}|)\sin\alpha + ...$$
$$+ (|a_1| - \tau|a_0|)\cos\alpha + (|a_1| + \tau|a_0|)\sin\alpha + (1-\tau)|a_0| + |a_0|]$$
$$\le R^{n+1}[(\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha$$
$$+ \lambda\sin\alpha + 1 - \lambda) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1} |a_j|]$$

for $R \ge 1$

and

$$\le |a_0| + R[(\rho + |a_n|)(\cos\alpha + \sin\alpha + 1) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha$$
$$+ \lambda\sin\alpha + 1 - \lambda) - \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1} |a_j|]$$

for $R \le 1$.

Hence the number of zeros of P(z) in $|z| \le \frac{R}{c} (R > 0, c > 1)$ does not exceed

$$\frac{1}{\log c} \log \frac{M_4}{|a_0|} ,$$

Where

$$M_4 = R^{n+1}[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha - \lambda + 1)$$
$$- \tau|a_0|(\cos\alpha - \sin\alpha + 1) + 2|a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1} |a_j|] \quad \text{for } R \ge 1$$

and

$$M_4 = |a_0| + R[(|\rho| + |a_n|)(\cos\alpha + \sin\alpha + 1) + |a_{n-k}|(\cos\alpha + \sin\alpha - \lambda\cos\alpha + \lambda\sin\alpha - \lambda + 1)$$
$$- \tau|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\ne n-k}^{n-1} |a_j|] \quad \text{for } R \le 1.$$

That completes the proof of Theorem 3.

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